O‘ZBEKISTON RESPUBLIKASI

TOG‘-KON SANOATI VA GEOLOGIYA VAZIRLIGI

NAVOIY DAVLAT KONCHILIK VA TEXNOLOGIYALAR UNIVERSITETI

“OLIY MATEMATIKA”

Bilim sohasi:	700000	–	Muhandislik, ishlov berish va qurilish sohalari
Ta’lim sohasi:	710100	–	Muhandislik ishi
Ta’lim yo‘nalishi:	60710400	–	Energetika muhandisligi
	60710500	–	Elektr muhandisligi
	60720800	–	Konchilik ishi
	60721200	–	Konchilik elektr mexanikasi
	61020200	–	Mehnat muhofazasi va texnika xavfsizligi
	60712300	–	Mexanika muhandisligi

NAVOIY-2025

“Oliy matematika” fanining o‘quv dasturiga asoslangan holda tayyorlangan ushbu o‘quv-uslubiy majmuada amaliy va nazariy mashg‘ulotlar keltirilgan. Mashg‘ulotlarning maqsadi talabalarni olgan bilimlarining nazariy qismini aniq injener-texnik masalalarni hal qilishda qo‘llashni o‘rgatish hisoblanadi. Har bir mashg‘ulotda berilgan masalalarni hal qilish uchun uslubiy ilovalari, hisoblashlar uchun berilgan qiymatlar, masalani yechish algoritmi, hisoblash formulalari, jadvallar, bajarilgan ishni topshirish tartibi va adabiyotlar ro‘yxati keltirilgan. Barcha masalalar namunalar bilan keltirilgan.

Tuzuvchi: S.I. Arziyeva - Navoiy davlat konchilik va texnologiyalari universiteti “Oliy matematika va axborot texnologiyalari” kafedrasi dotsenti v.b.

MUNDARIJA

KIRISH ...……		
1	Ko‘p o‘zgaruvchili murakkab funksiyaning hosilasi. Yuqori tartibli xususiy hosilalar va to‘la differensiallar. Ikki o‘zgaruvchili funksiyaning ekstremumi. Sirtga o‘tkazilgan urinma tekislik va normal tenglamasi…..	5
2	Birinchi tartibli differensial tenglamalar. O‘zgaruvchilari ajralgan va ajraladigan differensial tenglamalar……...	13
3	Bir jinsli differensial tenglamalar. Bir jinsli differensial tenglamaga keltiriladigan tenglamalar. Birinchi tartibli chiziqli differensial tenglamalar. Bernulli tenglamasi. To‘la differensialli tenglama..……..	17
4	Yuqori tartibli differensial tenglamalar. Tartibi pasaytiriladigan differensial tenglamalar……	27
5	O‘zgarmas koeffitsientli yuqori tartibli chiziqli bir jinsli differensial tenglamalar. O‘zgarmas koeffitsientli yuqori tartibli chiziqli bir jinsli bo‘lmagan, o‘ng tomoni maxsus ko‘rishishga ega bo‘lgan differensial tenglamalar……..…….....	30
6	Musbat hadli sonli qatorlar. Qator yig‘indisi. Qator yaqinlashishining zaruriy shartlari. Musbat hadli sonli qatorlarni taqqoslash...……..	39
7	Musbat hadli sonli qatorlar yaqinlashishining yetarli shartlari: Dalamber alomati, Koshining radikal va integral alomatlari. Ishorasi almashinuvchi va o‘zgaruvchan ishorali sonli qatorlar. Leybnits teoremasi. Absolyut va shartli yaqinlashish…….....	44
8	Funksional qatorlarning yaqinlashish sohasi. Darajali qatorlar. Yaqinlashish radiusi. Qatorlarni differensiallash va integrallash...……	51
9	Funksiyalarni Teylor va Makloren qatorlariga yoyish. Binomial qator. Asosiy elementar funksiyalarni qatorlarga yoyish. Fur’e qatori va Fur’e koeffitsiyentlari. Fur’e qatorining yaqinlashishi…..…….....	56
10	Ikki o‘lchovli integralni hisoblash. Ikki o‘lchovli integrallarning geometriya va mexanikaga tadbiqlariga doir mashqlar……	70
11	Uch o‘lchovli integralni hisoblash. Uch o‘lchovli integralning tadbiqlariga doir mashqlar…….	74
FOYDALANILGAN ADABIYOTLAR RO‘YXATI ……...		78

1-MAVZU. KO‘P O‘ZGARUVCHILI MURAKKAB FUNKSIYANING HOSILASI. YUQORI TARTIBLI XUSUSIY HOSILALAR VA TO‘LA DIFFERENSIALLAR. IKKI O‘ZGARUVCHILI FUNKSIYANING EKSTREMUMI. SIRTGA O‘TKAZILGAN URINMA TEKISLIK VA NORMAL TENGLAMASI

Agar $z = f(x,y), x = x(t), y = y(t)$ funksiyalar differensiallanuvchi bo‘lsa u holda $z = f(x(t), y(t))$ murakkab funksiyaning hosilasi ushbu formula bo‘yicha topiladi:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \tag{1}$$

Agar $z = f(x,y), y = y(x)$ bo‘lsa, u holda $z = f(x, y(x))$ dan x bo‘yicha to‘liq hosila quyidagi formula bo‘yicha topiladi:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \tag{2}$$

Xuddi shuningdek, $x = x(u,v), y = y(u,v)$ bo‘lsa, u holda $z = f(x,y)$ ning xususiy hosilalari formulalar bo‘yicha topiladi:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}, \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \tag{3}$$

1-misol. Agar $x = e^t$ va $y = lnt$ bo‘lsa, $z = \frac{x}{y}$ funksiyaning hosilasini toping.

Yechish.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \frac{e^t}{y} - \frac{x}{y^2} \cdot \frac{1}{t} = \frac{tye^t - x}{y^2 \cdot t}$$

2-misol. Agar $y = x^2$ bo‘lsa, $z = arctg\frac{y}{x}$ funksiyaning to‘liq hosilasini toping.

Yechish. To‘liq hosila formulasiga ko‘ra:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} \cdot 2x = \frac{-y}{x^2+y^2} + \frac{2x^2}{x^2+y^2} = \frac{2x^2-y}{x^2+y^2} =$$

$$\frac{2x^2-x^2}{x^2+x^4} = \frac{1}{1+x^2}.$$

Xususiy hosila: $\frac{\partial z}{\partial x} = -\frac{y}{x^2+y^2}$

3-misol. Agar $x = uv, y = \frac{u}{v}$ bo‘lsa, $z = \ln(x^2 + y^2)$ funksiyaning xususiy hosilalarini toping.

Yechish. $\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=\frac{2x}{x^2+y^2}\cdot v+\frac{2y}{x^2+y^2}\cdot\frac{1}{v}=\frac{2}{x^2+y^2}\left(x\cdot v+\frac{y}{v}\right)=\frac{2}{u}$; $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{2x}{x^2+y^2}\cdot u+\frac{2y}{x^2+y^2}\cdot\left(-\frac{u}{v^2}\right)=\frac{2}{x^2+y^2}\left(xu-\frac{yu}{v^2}\right)=\frac{2(v^4-1)}{(v^4+1)v}$.

Agar $F(x,y,z)=0$ tenglama ikki o‘zgaruvchili $z(x,y)$ funksiyani oshkormas ko‘rinishda aniqlasa va $F_z'(x,y,z)\neq 0$ bo‘lsa, u holda quyidagi formulalar o‘rinlidir:

$$\frac{\partial z}{\partial x}=-\frac{F_x'(x,y,z)}{F_z'(x,y,z)},\qquad \frac{\partial z}{\partial y}=-\frac{F_y'(x,y,z)}{F_z'(x,y,z)}$$

4-misol. Oshkormas ko‘rinishda

$$(x^2+y^2)^3-3(x^2+y^2)+1=0$$

tenglama bilan berilgan $y(x)$ funksiyaning hosilasini toping.

Yechish. Tenglamaning chap tomonini $F(x,y)$ orqali belgilaymiz va xususiy hosilalarni topamiz:

$$F_x'(x,y)=3(x^2+y^2)^2 2x-6x=((x^2+y^2)^2-1)\cdot 6x$$
$$F_y'(x,y)=3(x^2+y^2)^2 2y-6y=((x^2+y^2)^2-1)\cdot 6y$$

Demak, $\frac{dy}{dx}=-\frac{F_x'(x,y)}{F_y'(x,y)}=-\frac{6x((x^2+y^2)^2-1)}{((x^2+y^2)^2-1)\cdot 6y}=-\frac{x}{y}$

5-misol. Oshkormas ko‘rinishda $x^2-2y^2+3z^2-yz+y=0$ tenglama bilan berilgan $z(x,y)$ funksiyaning xususiy hosilalarini toping.

Yechish. Tenglamaning chap tomonini $F(x,y,z)$ orqali belgilab, xususiy hosilalarni topamiz:

$$F_x'=(x,y,z)=2x;\ F_y'(x,y,z)=-4y-z+1;\ F_z'(x,y,z)=6z-y.$$

Demak,

$$\frac{\partial z}{\partial x}=-\frac{F_x'(x,y,z)}{F_z'(x,y,z)}=-\frac{2x}{6z-y};\ \frac{\partial z}{\partial y}=-\frac{F_y'(x,y,z)}{F_z'(x,y,z)}=-\frac{1-4y-z}{6z-y}.$$

Sirtga M_0 nuqtada o‘tkazilgan urinma tekislik deb sirtda M_0 nuqta orqali o‘tkazilgan barcha egri chiziqlarga o‘tkazilgan urinmalar joylashgan tekislikka aytiladi. Sirtga M_0nuqtadagi normal deb M_0 nuqtadan o‘tuvchi va bu nuqtada o‘tkazilgan urinma tekislikka perpendikular bo‘lgan to‘g’ri chiziqqa aytiladi.

Agar sirt $z = f(x, y)$ tenglama bilan berilgan bo‘lsa, u holda $M_0(x_0, y_0, z_0)$ nuqtada bu sirtga o‘kazilgan urinma tekislik tenglamasi:

$$z - z_0 = f_x'(x_0, y_0)(x - x_0) + f_y'(x_0, y_0)(y - y_0). \quad (4)$$

$M_0(x_0, y_0, z_0)$ nuqta orqali sirtga o‘tkazilgan normalning kanonik tenglamasi quyidagicha bo‘ladi:

$$\frac{x-x_0}{f_x'(x_0,y_0)} = \frac{y-y_0}{f_y'(x_0,y_0)} = \frac{z-z_0}{-1}. \quad (5)$$

Agar sirt $F(x, y, z) = 0$ tenglama bilan oshkormas ko‘rinishda berilgan bo‘lsa, sirtning $M_0(x_0, y_0, z_0)$ nuqtasida o‘tkazilgan urinma tekislik tenglamasi esa $\frac{x-x_0}{F_x'(x_0,y_0,z_0)} = \frac{y-y_0}{F_y'(x_0,y_0,z_0)} = \frac{z-z_0}{F_z'(x_0,y_0,z_0)}$ ko‘rinishda bo‘ladi.

6-misol. $z = x^2 - 2xy + y^2 - x + 2y$ sirtga $M_0(1,1,1)$ nuqtada o‘tkazilgan urinma tekislik va normal tenglamalarini tuzing.

Yechish. Xususiy hosilalarni topamiz:

$$\frac{\partial z}{\partial x} = 2x - 2y - 1 \text{ va } \frac{\partial z}{\partial y} = -2x + 2y + 2.$$

Bu hosilalarning $M_0(1,1,1)$ nuqtadagi qiymatlarini hisoblaymiz:

$\frac{\partial z}{\partial x}|M_0 = 2 - 2 - 1 = -1$ va $\frac{\partial z}{\partial x}|M_0 = -2 + 2 + 2 = 2.$

Shunday qilib, $f_x'(1,1) = -1, f_x'(1,1) = 2$. Demak, urinma tekislik tenglamasi:

$z - 1 = -(x - 1) + 2(y - 1)$ yoki $x - 2y + z = 0$,

normal tenglamasi:

$$\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-1}{1}.$$

7-misol. $x^3 + y^3 + z^3 + xyz - 6 = 0$ sirtga $M_0(1,2,-1)$ nuqtada o‘tkazilgan urinma tekislik va normalning tenglamasini tuzing.

Yechish. Tenglamaning chap tomonini $F(x, y, z)$ orqali belgilab, xususiy hosilalarni topamiz:

$$F_x'(x, y, z) = 3x^2 + yz; F_y'(x, y, z) = 3y^2 + xz;$$

$$F_z'(x, y, z) = 3z^2 + xy;$$

Hosilalarning $M_0(1,2,-1)$ nuqtadagi qiymatlarini hisoblaymiz:

$$F_x'(1,2,-1) = 3 - 2 = 1;\ F_y'(1,2,-1) = 12 - 1 = 11;$$

$$F_z'(1,2,-1) = 3 + 2 = 5.$$

Shunday qilib, urinma tekislik tenglamasi:

$1(x-1) + 11(y-2) + 5(z+1) = 0$ yoki $x + 11y + 5z - 18 = 0$,

$z = f(x,y)$ funksiyaning ikkinchi tartibli xususiy hosilalari deb birinchi tartibli xususiy hosilalardan olingan xususiy hosilalarga aytiladi. Ikkinchi tartibli xususiy hosilalar quyidagicha belgilanadi:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = z''_{xx} = f''_{xx}(x,y);$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = z''_{xy} = f''_{xy}(x,y);$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = z''_{yx} = f''_{yx}(x,y);$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = z''_{yy} = f''_{yy}(x,y).$$

$f''_{xy}(x,y)$ va $f''_{yx}(x,y)$ xususiy hosilalar aralash hosilalar deyiladi. Aralash hosilalar uzluksiz bo‘lgan nuqtalarda ularning qiymatlari teng bo‘ladi. Uchinchi va undan yuqori tartibli xususiy hosilalar ham shunday aniqlanadi. Ushbu $\frac{\partial^n z}{\partial x^m \partial y^{n-m}}$ yozuv z funksiyani m marta x o‘zgaruvchi bo‘yicha va $(n-m)$ marta y o‘zgaruvchi bo‘yicha differensiallanganini bildiradi.

Agar $z = f(x,y)$ funksiyaning $P_0(x_0, y_0)$ nuqtadagi qiymati uning bu nuqtaning biror atrofidagi istalgan $P(x,y)$ nuqtasidagi qiymatidan katta, ya’ni $f(x_0, y_0) > f(x,y)$ bo‘lsa, $z = f(x,y)$ funksiya $P_0(x_0, y_0)$ nuqtada maksimumga ega deyiladi.

Agar $z = f(x,y)$ funksiyaning $P_0(x_0, y_0)$ nuqtadagi qiymati uning bu nuqtaning biror atrofidagi istalgan $P(x,y)$ nuqtasidagi qiymatidan kichik, ya’ni

$f(x_0, y_0) < f(x, y)$ bo‘lsa, $z = f(x, y)$ funksiya $P_0(x_0, y_0)$ nuqtada minimumga ega deyiladi.

Funksiyaning maksimumi yoki minimumi uning ekstremumi deyiladi. Funksiya ekstremumga ega bo‘lgan nuqta uning ekstremum nuqtasi deyiladi. Ekstremumning zaruriy shartlari: agar $P_0(x_0, y_0)$ nuqta uzluksiz $z = f(x, y)$ funksiyaning ekstremum nuqtasi bo‘lsa, u holda $f'_x(x_0, y_0) = f'_y(x_0, y_0) = 0$ bo‘ladi yoki bu hosilalarning aqalli bittasi mavjud bo‘lmaydi.

Bu shartlar bajariladigan nuqtalar kritik nuqtalar deyiladi. Har qanday kritik nuqta ham ekstremum nuqta bo‘lavermaydi.

Ikkinchi tartibli hosilalarning $P_0(x_0, y_0)$ kritik nuqtadagi qiymatlarini

$$A = f''_{xx}(x_0, y_0);\ B = f''_{xy}(x_0, y_0);\ C = f''_{yy}(x_0, y_0)$$

Orqali belgilaymiz va $\Delta = AC - B^2$ diskriminantni tuzamiz.

Ekstremumning yetarli sharti.

a) Agar $\Delta > 0$ bo‘lsa, $z = f(x, y)$ funksiya $P_0(x_0, y_0)$ nuqtada ekstremumga ega bo‘lib, bunda $A < 0$ (yoki $C < 0$) bo‘lganda P_0 nuqta maksimum nuqtasi, $A > 0$ (yoki $C > 0$) bo‘lganda minimum nuqtasi bo‘ladi;
b) Agar $\Delta < 0$ bo‘lsa, P_0 nuqtada ekstremum mavjud emas;
c) Agar $\Delta = 0$ bo‘lsa, ekstremum mavjud bo‘lishi ham mavjud bo‘lmasligi ham mumkin.

8-misol. $z = xy(x + y - 2)$ funksiyaning ekstremumlarini toping.

Yechish. Funksiya butun Oxy tekislikda aniqlangan.

Kritik nuqtalarni quyidagi tenglamalardan topamiz:

$$\frac{\partial z}{\partial x} = 2xy + y^2 - 2y = 0,$$

$$\frac{\partial z}{\partial y} = 2xy + x^2 - 2x = 0$$

Bu sistemani yechib, to‘rtta kritik nuqtani topamiz: $P_2(2,0)$, $P_3(0,2)$, $P_4(\frac{2}{3}, \frac{2}{3})$.

Ikkinchi tartibli xususiy hosilalar.

$$f''_{xx}(x, y) = 2y;\ f''_{xy}(x, y) = 2x + 2y - 2;\ f''_{yy}(x, y) = 2x.$$

Har bir kritik nuqtadagi diskriminantni hisoblaymiz:

a) $P_1(0,0)$, nuqtada: $\Delta = AC - B^2 = -4 < 0$, demak ekstremum yo‘q.

b) $P_2(2,0)$, nuqtada: $\Delta = -4 < 0$, demak ekstremum mavjud emas;

c) $P_3(0,2)$, nuqtada $\Delta = -4 < 0$, demak ekstremum mavjud emas;

d) $P_4(\frac{2}{3},\frac{2}{3})$ nuqtada: $\Delta = \frac{12}{9} > 0, A = \frac{4}{3} > 0$, demak funksiyaning minimum nuqtasiga egamiz, bu nuqtada $z_{min} = f\left(\frac{2}{3}, \frac{2}{3}\right) = -\frac{8}{27}$.

Chegaralangan yopiq $\overline{D}$ sohada differensiallanuvchi funksiya o‘zining eng katta va eng kichik qiymatiga yoki $\overline{D}$ soha ichida yotuvchi kritik nuqtada, yoki bu soha chegarasida erishadi.

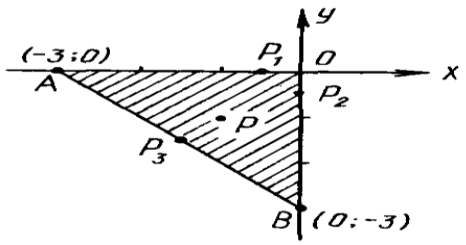
Yopiq $\overline{D}$ sohada funksiya eng katta va eng kichik qiymatini topish uchun: a) soha ichida va uning chegarasida yotgan barcha kritik nuqtalar topiladi; b) funksiyaning bu nuqtalardagi va chegaradagi qiymatlari hisoblanadi; c) topilgan qiymatlar orasidan eng katta va eng kichik qiymatlar topiladi.

9-misol. $z = x^2 + y^2 - xy + x + y$ funksiyaning $x \le 0, y \le 0, x + y \ge -3$ sohadagi eng katta va eng kichik qiymatlarini toping.

Yechish. D soha AOB uchburchakdan iborat (1-rasm).

a) Ushbu sistemadan soha ichidagi kritik nuqtalarni topamiz:

$$\begin{cases} \frac{\partial z}{\partial x} = 2x - y + 1 = 0; \\ \frac{\partial z}{\partial y} = 2y - x + 1 = 0. \end{cases}$$



Bu yerdan $x = -1, y = -1$, demak $P_0(-1,-1)$ nuqtaga egamiz.

b) Funksiyani soha chegarasida tekshiramiz. Tenglamasi $y = 0$ bo‘lgan AO chegarada $z = x^2 + x$ funksiyaga egamiz: kritik nuqtalarning abssissalarini $z'_x = 2x + 1 = 0$ tenglamadan aniqlaymiz: $x = -\frac{1}{2}$. Demak kritik nuqta: $P_1(-\frac{1}{2}, 0)$. Tenglamasi $x = 0$ bo‘lgan BO chegarada $z = y^2 + y$ funksiyaga egamiz: kritik

nuqtalarning ordinalarini $z'_y = 2y + 1 = 0$ tenglamadan topamiz: $y = -\frac{1}{2}$. Demak, kritik nuqta $P_2\left(0, -\frac{1}{2}\right)$. Tenglamasi $y = -3 - x$

bo‘lgan AB chegarada $z = 3x^2 + 9x + 6$ funksiyaga egamiz; kritik nuqtalarning abssissalarini $z'_x = 6x + 9 = 0$ tenglamadan topamiz: $x = -\frac{3}{2}$. AB tenglamasidan $y = -\frac{3}{2}$. Demak kritik nuqta:

$$P_3(-\tfrac{3}{2}, -\tfrac{3}{2}).$$

c) Berilgan funksiyaning P_0, P_1, P_2, P_3 kritik nuqtalardagi hamda chegaralar tutashadigan A, B va O nuqtalardagi qiymatlarini hisoblaymiz:

$$z_0 = f(P_0) = -1;$$

$$z_1 = f(P_1) = -\frac{1}{4};$$

$$z_2 = f(P_2) = -\tfrac{1}{4};$$

$$z_3 = f(P_3) = -\frac{3}{4};$$

$$z_4 = f(O) = 0$$

$$z_5 = f(A) = 6$$

$$z_6 = f(B) = 6$$

d) Funksiyaning barcha topilgan qiymatlarini taqqoslab, $z_{eng\ kat.} = f(A) = f(B) = 6$ va $z_{eng\ kich.} = f(P_0) = -1$ degan xulosaga kelamiz.

Topshiriqlar.

1. Agar $z = ln\frac{u}{v}$, bu yerda $u = tg^2x$, $v = ctg^2x$ bo‘lsa, $\frac{dz}{dx}$ ni toping.
2. Agar $z = \frac{x^2-y}{x^2+y}$, bu yerda $y = 3x + 1$ bo‘lsa, $\frac{dz}{dx}$ ni toping.
3. Agar $z = x^2y$, bu yerda $y = cosx$ bo‘lsa, $\frac{\partial z}{\partial x}$ va $\frac{dz}{dx}$ ni toping.
4. Agar $z = u^2lnv$, bu yerda $u = \frac{y}{x}$, $v = x^2 + y^2$ bo‘lsa, $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ ni toping.
5. Agar $z = x^2y - y^2x$, bu yerda $x = ucosv$ va $y = usinv$ bo‘lsa, $\frac{\partial z}{\partial u}$ va $\frac{\partial z}{\partial v}$ ni toping.

6. Agar $w = \ln(x^3 + y^3 - z^3)$, bu yerda $x = uv, y = \frac{u}{v}$, $z = e^{uv}$ bo‘lsa, $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$ ni toping.

7. Oshkormas ko‘rinishda $sinxy - e^{xy} - x^2y = 0$, tenglama bilan berilgan $y(x)$ funksiyaning hosilasini toping.

8. Oshkormas ko‘rinishda $e^z = xyz$ tenglama bilan berilgan $z(x, y)$ funksiyaning hosilasini toping.

9. Oshkormas ko‘rinishda $y^x = x^y$ tenglama bilan berilgan $y(x)$ funksiyaning hosilasini toping.

10. Oshkormas ko‘rinishda $z^3 + 3xyz = a^3$ tenglama bilan berilgan $z(x, y)$ funksiyaning hosilasini toping.

11. Agar $z = \arcsin(x - y)$, bu yerda $x = 3t, y = 4t^3$ bo‘lsa, $\frac{dz}{dt}$ ni toping.

12. Agar $z = arctgxy$, bu yerda $y = e^x$ bo‘lsa, $\frac{dz}{dx}$ ni toping.

13. Agar $z = v^2 + u^2$, bu yerda $u = x + y, v = x - y$ bo‘lsa, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ larni toping.

14. Agar $z = u^2v - v^2u$, bu yerda $u = xsiny, v = ycosx$ bo‘lsa, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ larni toping.

Berilgan sirtga berilgan $M_0(x_0, y_0, z_0)$ nuqtada o‘tkazilgan urinma tekislik va normal tenglamalarini tuzing.

15. $z = 1 + x^2 + y^2$, $M_0(1,1,3)$

16. $x^2 + z^2 - 5yz + 3y = 46$, $M_0(1,2,-3)$.

Berilgan funksiyalarning ikkinchi tartibli xususiy hosilalarni toping va aralash xususiy hosilalari teng yoki teng emasligini tekshiring:

17. $z = xy + \frac{y}{x}$;

18. $z = xe^{-xy}$

19. Funksiyani ekstremumga tekshiring. $z = x^2 + y^2 + xy - 3x - 6y$;

20. Funksiyaning ko‘rsatilgan sohadagi eng katta va eng kichik qiymatlarini toping. $z = x^2 + y^2 - xy - 4x; x \geq 0, y \geq 0, 2x + 3y \leq 12$;

2-MAVZU. BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMALAR. O‘ZGARUVCHILARI AJRALGAN VA AJRALADIGAN DIFFERENSIAL TENGLAMALAR

Ta’rif: Differensial tenglama deb erkli o‘zgaruvchi x, noma’lum $y = f(x)$ funksiya va uning $y', y'', \ldots, y^{(n)}$ hosilalari orasidagi bog‘lanishni ifodalaydigan tenglamaga aytiladi. Difrensial tenglamani simvolik ravishda quyidagicha yozish mumkin:

$$F\left(x, y, y' y'', \ldots, y^{(n)}\right) = 0 \text{ yoki}$$

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^4y}{dx^4}\right) = 0$$

Agar izlangan funksiya $y = f(x)$ bitta erkli o‘zgaruvchining funksiyasi bo‘lsa, u holda differensial tenglama oddiy defrensial tenglama deyiladi. Differensial tenglamaning tartibi deb tenglamaga kirgan hosilaning eng yuqori tartibiga aytiladi.Birinchi tartibli differensial tenglama

$$F\left(x, y, y'\right) = 0 \quad \text{yoki} \quad y' = f(x, y)$$

ko‘rinishda bo‘ladi.

Teorema: Agar $y' = f(x, y)$ tenglamadagi $f(x, y)$ funksiya va undan y bo‘yicha olingan $\frac{\partial f}{\partial y}$ xususiy hosila XOY tekislikdagi (x_0, y_0) nuqtani o‘z ichiga oluvchi biror sohada uzluksiz funksiyalar bo‘lsa, u holda berilgan tenglamaning $x = x_0$ bo‘lganda $y = y_0$ shartni qanoatlantiruvchi birgina $y = \varphi(x)$ yechimi mavjuddir.

$x = x_0$ bo‘lganda y funksiya berilgan y_0 songa teng bo‘lishi kerak degan shart boshlang‘ich shart deyiladi.

Diffrensial tenglamalardan biri o‘zgaruvchilari ajralgan va o‘zgaruvchilari ajraladigan difrensial tenglamalardir.

1. $M(x)dx+N(y)dy=0$ - o‘zgaruvchilari ajralgan difrensial tenglama deyiladi. Bu tenglamaning umumiy integrali $\int M(x)dx+\int N(y)dy=C$ - ko‘rinishda bo‘ladi.

2. Ushbu $M_1(x)N_1(y)dx+M_2(x)N_2(y)dy=0$ - ko‘rinishdagi tenglama deyiladi. Bu tenglamaning ikkala tomonini $N_1(y)\cdot M_2(x)$ ifodaga bo‘lish bilan o‘zgaruvchilari ajralgan tenglamaga keltirish mumkin, ya`ni

$$\frac{M_1(x)}{M_2(x)}dx+\frac{N_2}{N_1}dy=0$$

1- va 2- ko‘rinishdagi tenglamalarga doir misollar ko‘ramiz.

Izlanayotgan xususiy yechim quyidagicha bo‘ladi:

$$\left(1+y^3\right)^2-4\left(1+x^2\right)^3=0.$$

Yuqorida $\sqrt{1-y^2}$ ga bo‘lish jarayonida $y=\pm1$ ikkita yechim yo‘qolganini qayd etamiz. $y=\pm1$ yechimlarni bevosita tekshiraylik:

$$y=1\ \sqrt{1-1^2}dx+\sqrt{1-x^2}d1=0\ dx+\sqrt{1-x^2}\ 0=0\,.$$

$y=-1$ ham shu kabi tekshiriladi. Demak, $y=-1$ va $y=1$ lar ham berilgan tenglamaning yechimlari ekan. $y\big|_{x=0}=0$ boshlang’ich shartni e’tiborga olsak, $c_1=0$ hosil bo‘ladi. U holda, berilgan tenglamaning xususiy yechimlaridan biri quyidagicha ko‘rinishga ega:

$$x\sqrt{1-y^2}+y\sqrt{1-x^2}=0.$$

3. $e^{-s}\left(1+\dfrac{ds}{dt}\right)=1$

<u>Yechish</u>: Tenglamani har ikkala tomonini e^s ga ko‘paytiramiz:

$1+\dfrac{ds}{dt}=e^s\ \ \dfrac{ds}{dt}=e^s-1\Rightarrow ds=\left(e^s-1\right)dt\Rightarrow\ \dfrac{ds}{e^s-1}=dt$ tenglamani integrallasak,

$$\int \frac{ds-1}{e^s} = \int dt$$

$\int \frac{ds}{e^s-1}$ integralni topish uchun $e^s = x$ almashtirish qilamiz, $s = \ell nx$ dan $ds = \frac{dx}{x}$ ni topamiz, u holda integral $\int \frac{dx}{x(x-1)}$ ko‘rinishda bo‘ladi.

$$\int \frac{x-(x-1)}{x(x-1)}dx = \int \frac{dx}{x-1} - \int \frac{dx}{x} = \ell n|x-1| - \ell n|x| = \ell n\left(1-\frac{1}{x}\right)$$

Oxirgi ifodani tenglikni chap qismiga qo‘ysak, u holda $\ell n\left(1-\frac{1}{x}\right) = t + C$ bo‘ladi.

$e^s = x$ almashtirishni hisobga olsak,

$$\ell n\left(1-\frac{1}{e^s}\right) = t + C \Rightarrow \ell n\left(1-\frac{1}{e^s}\right) + \ell nC = t \Rightarrow$$
$$\left(1-e^{-s}\right)C = e^t$$

umumiy javobga ega bo‘lamiz.

Topshiriqlar.

Differensial tenglamaning umumiy yechimini toping:

1. $\left(xy^2 + x\right)dx + \left(y - x^2 y\right)dy = 0$

2. $xyy^1 = 1 - x^2$

3. $y' \cdot tgx - y = a$

4. $y' + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

5. $y' + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$

6. $y'\sin x = y\ell ny$; $y_{x=\frac{\Pi}{2}} = 1$, boshlang‘ich shartni qanoatlantiruvchi yechimni toping.

7. $y' = \frac{1+y^2}{1+x^2}$; $y_{x=0} = 1$;

8. $y - xy' = b\left(1 + x^2 y\right)$; $y_{x=1} = 1$

9. $y' = \frac{y^2}{x^2} - 2$

10. $xdy - ydx = ydy$

11. $tgxdx - ctgxdy = 0$ tenglamaning umumiy yechimini toping.

12. Ushbu $(1 + x^2)dy + ydx = 0$ tenglamaning $y|_{x=1} = 1$, boshlang'ich shartni qanoatlantiruvchi yechimini toping.

Differensial tenglamaning umumiy yechimini toping:

13. $(1 + x^2)y' + 1 + y^2 = 0$

14. $(x + x^2)y' = 2y + 1$

Koshi masalasini yeching.

15. $sec^2 xtgydx + sec^2 ytgxdy = 0;\ y|_{x=\frac{\pi}{4}} = \frac{\pi}{4}$

16. $xy' - y = xtg\frac{y}{x};\ \ y|_{x=1} = \frac{\pi}{2}$

17. $\left(\sqrt{xy} - \sqrt{x}\right)dx + \left(\sqrt{xy} + \sqrt{y}\right)dy = 0$

18. $xyy' = y^2 + 2x^2$

19. $xy + y^2 = (2x^2 + xy)y'$

20. $xy' + 2\sqrt{xy} = y$

3-MAVZU. BIR JINSLI DIFFERENSIAL TENGLAMALAR. BIR JINSLI DIFFERENSIAL TENGLAMAGA KELTIRILADIGAN TENGLAMALAR. BIRINCHI TARTIBLI CHIZIQLI DIFFERENSIAL TENGLAMALAR. BERNULLI TENGLAMASI. TO‘LA DIFFERENSIALLI TENGLAMA

Bir jinsli differensial tenglama tushunchasini yoritish uchun, oldin bir jinsli funksiya tushunchasini kiritamiz.

Ta’rif: Agar $f(x,y)$ funksiya ixtiyoriy o‘zgarmas λ soni uchun

$$f(\lambda x, \lambda y) = f(x, y)$$

shartni qanoatlantirsa, bu funksiya x va y o‘zgaruvchilarga nisbatan ***bir jinsli funksiya*** deb ataladi.

Masalan,

$$f(x,y) = \frac{x-y}{\sqrt{x^2+y^2}}$$

bir jinsli funksiya bo‘ladi, chunki

$$f(\lambda x, \lambda y) = \frac{\lambda x - \lambda y}{\sqrt{(\lambda x)^2 + (\lambda y)^2}} = \frac{\lambda(x-y)}{\lambda\sqrt{x^2+y^2}} = \frac{x-y}{\sqrt{x^2+y^2}} = f(x,y).$$

Xuddi shunday tarzda

$$f(x,y) = \sin\frac{y}{x}, \quad f(x,y) = \frac{x-y}{x+y}, \quad f(x,y) = \frac{xy}{x^2+y^2}$$

funksiyalar ham bir jinsli bo‘lishini ko‘rsatish mumkin va buni o‘quvchiga havola etamiz.

Lemma: Agar $f(x,y)$ bir jinsli funksiya bo‘lsa, uni $f(x,y)=g(y/x)$ ko‘rinishda yozish mumkin.

Isbot: Funksiyaning bir jinslilik shartida $\lambda=1/x$ deb olib

$$f(x,y) = f\left(\frac{1}{x}\cdot x, \frac{1}{x}\cdot y\right) = f\left(1, \frac{y}{x}\right) = g\left(\frac{y}{x}\right)$$

lemma tasdig‘iga ega bo‘lamiz.

Masalan,

$$f(x,y)=\frac{x-y}{\sqrt{x^2+y^2}}=\frac{x(1-y/x)}{x\sqrt{1+(y/x)^2}}=\frac{1-y/x}{\sqrt{1+(y/x)^2}}=g(\frac{y}{x}).$$

Ta'rif: Agar birinchi tartibli

$$y'=f(x,y)$$

tenglamada $f(x,y)$ bir jinsli funksiya bo'lsa, u ***bir jinsli differensial tenglama*** deyiladi.

Lemmaga asosan bir jinsli I tartibli differensial tenglamani

$$y'=g(\frac{y}{x}) \qquad (1)$$

ko'rinishda yozish mumkin. Bu tenglamani integrallash uchun $u(x)=u=y/x$ almashtirma bajaramiz. Bu holda

$$y=ux\Rightarrow y'=(ux)'=u'x+u$$

tenglikka ega bo'lamiz. Bu tenglik va (1) tenglamadan foydalanib $u=u(x)$ funksiya uchun ushbu tenglamani hosil etamiz:

$$y'=g(\frac{y}{x})\Rightarrow u'x+u=g(u)\Rightarrow x\frac{du}{dx}=g(u)-u\Rightarrow xdu=(g(u)-u)dx.$$

Bu o'zgaruvchilari ajraladigan differensial tenglama bo'lib, uning umumiy integralini yuqorida ko'rsatilgan usulda topamiz:

$$xdu=(g(u)-u)dx\Rightarrow\frac{du}{g(u)-u}=\frac{dx}{x}\Rightarrow\int\frac{du}{g(u)-u}=\int\frac{dx}{x}=\ln|x|+C \qquad (2)$$

(2) tenglamadan $u=u(x,C)$ umumiy yechimni aniqlagach, berilgan (1) tenglamaning umumiy integralini $y=xu$ formula orqali topamiz.

Misol sifatida bir jinsli

$$y'=\frac{x+y}{x}$$

differensial tenglamani ko'rsatilgan usulda integrallaymiz:

$$y'=\frac{x+y}{x}\Rightarrow(ux)'=\frac{x+ux}{x}\Rightarrow u'x+u=1+u\Rightarrow u'=\frac{1}{x}\Rightarrow$$

$$\Rightarrow u=\int\frac{dx}{x}=\ln|x|+C\Rightarrow y=xu=x\ln|x|+Cx.$$

Ta`rif: Agar λ ning har qanday qiymatida $f(\lambda x,\lambda y)=\lambda^n f(x,y)$ ayniyat to‘g‘ri bo‘lsa, $f(x,y)$ funksiya x va y o‘zgaruvchilarga nisbatan n-o‘rinli bir jinsli funksiya deyiladi.

1-misol. $f(x,y)=\sqrt[3]{x^3+y^3}$ funksiya bir o‘lchovli bir jinsli funksiya, chunki

$$f(\lambda x,\lambda y)=\sqrt[3]{(\lambda x)^3+(\lambda y)^3}=\sqrt[3]{\lambda^3\left(x^3+y^3\right)}=\lambda\sqrt[3]{x^3+y^3}=\lambda\cdot f(x,y);$$

2-misol. $f(x,y)=xy-y^2$ -funksiya ikki o‘lchamli bir jinsli funksiya, chunki

$$f(\lambda x,\lambda y)=(\lambda x)\cdot(\lambda y)-(\lambda y)^2=\lambda^2(xy)-\lambda^2y^2=\lambda^2\left(xy-y^2\right)=\lambda^2 f(x,y)$$

3-misol. $f(x,y)=\dfrac{x^2-y^2}{xy}$ - nol o‘lchovli bir jinsli funksiya, chunki

$$f(\lambda x,\lambda y)=\frac{(\lambda x)^2-(\lambda y)^2}{(\lambda x)(\lambda y)}=\frac{\lambda^2\left(x^2-y^2\right)}{\lambda^2xy}=\frac{x^2-y^2}{xy}=f(x,y)=\lambda^0 f(x,y)$$

I tartibli chiziqli differensial tenglama. Bu tenglamada noma’lum funksiya y va uning hosilasi y' birinchi darajada, ya’ni chiziqli ravishda qatnashib, quyidagi ko‘rinishda bo‘ladi:

$$y'+P(x)y=Q(x) \tag{3}$$

Bunda $P(x)$ va $Q(x)$ uzluksiz funksiyalar yoki o‘zgarmas sonlardir. Agar $Q(x)\equiv0$ bo‘lsa, (3) bir jinsli , aks holda bir jinsli bo‘lmagan chiziqli tenglama deyiladi.

(3) chiziqli tenglamani Bernulli usulida yechamiz. Buning uchun umumiy yechimni $y=u(x)v(x)=uv$ ko‘rinishda izlaymiz. Bu yerda $u(x)$ va $v(x)$ noma’lum funksiyalar bo‘lib, ularni topish uchun (3) tenglamada y o‘rniga uv ko‘paytmani qo‘yib, ushbu tenglamaga ega bo‘lamiz:

$$(uv)'+P(x)uv=Q(x)\Rightarrow u'v+uv'+P(x)uv=Q(x)\Rightarrow$$

$$u'v+u[v'+P(x)v]=Q(x) \tag{4}$$

Oxirgi, ya’ni (4) tenglamadan, kvadtrat qavs ichidagi ifodani nolga tenglashtirib, $v=v(x)$ noma’lum funksiya uchun

$$v'+P(x)v=0 \qquad (**)$$

tenglamaga ega bo‘lamiz. Bunda (**) berilgan bir jinsli bo‘lmagan (3) chiziqli tenglamaga mos keluvchi bir jinsli tenglama ekanligini ta’kidlab o‘tamiz. Bu

o‘zgaruvchilari ajraladigan differensial tenglama bo‘lib, uni yuqorida ko‘rilgan usulda yechamiz:

$$v' + P(x)v = 0 \Rightarrow v' = -P(x)v \Rightarrow \frac{dv}{dx} = -P(x)v \Rightarrow \frac{dv}{v} = -P(x)dx \Rightarrow$$

$$\Rightarrow \int \frac{dv}{v} = -\int P(x)dx \Rightarrow \ln|v| - C = -\int P(x)dx \Rightarrow v = e^{C-\int P(x)dx} = C_1 e^{-\int P(x)dx}$$

Bu yerda C_1=1 deb, izlanayotgan noma’lum funksiyalardan biri

$$v(x) = e^{-\int P(x)dx} \tag{5}$$

ekanligini ko‘ramiz. Bu natijani (4) tenglamaga qo‘yib va (**) tenglikdan foydalanib, ikkinchi $u=u(x)$ noma’lum funksiyani topamiz:

$$u'v + u \cdot 0 = Q(x) \Rightarrow u' = \frac{Q(x)}{v(x)} \Rightarrow u(x) = \int \frac{Q(x)}{v(x)} dx + C \tag{6}$$

Bu yerdan berilgan (3) chiziqli differensial tenglamaning umumiy integrali

$$y = uv = e^{-\int P(x)dx}[\int e^{\int P(x)dx} Q(x)dx + C] \tag{7}$$

formula bilan topilishini ko‘ramiz.

Misol sifatida ushbu Koshi masalasini yechamiz:

$$y' + 2xy = xe^{-x^2}, \quad y\big|_{x=0} = 3.$$

Dastlab Koshi masalasidagi I tartibli chiziqli differensial tenglamaning umumiy yechimini Bernulli usulida topamiz, ya’ni $y=u\cdot v$ ko‘rinishda izlaymiz.

$$(uv)' + 2xuv = xe^{-x^2} \Rightarrow u'v + uv' + 2xuv = xe^{-x^2} \Rightarrow u'v + u[v' + 2xv] = xe^{-x^2}$$

Kvadrat qavs ichidagi ifodani nolga tenglashtirib, $v=v(x)$ funksiyani topamiz:

$$v' + 2xv = 0 \Rightarrow \frac{dv}{dx} = -2xv \Rightarrow \frac{dv}{v} = -2xdx \Rightarrow \int \frac{dv}{v} = -\int 2xdx \Rightarrow v = Ce^{-x^2}.$$

Demak, $v(x) = e^{-x^2}$ deb olish mumkin. Unda

$$u'v + u[v' + 2xv] = xe^{-x^2} \Rightarrow u'e^{-x^2} = xe^{-x^2} \Rightarrow u' = x \Rightarrow u(x) = \int xdx = \frac{x^2}{2} + C.$$

Bu yerdan berilgan differensial tenglamaning umumiy yechimi

$$y = uv = (\frac{x^2}{2} + C)e^{-x^2}$$

ko‘rinishda ekanligi kelib chiqadi. Undagi C o‘zgarmas sonini topish uchun Koshi masalasining boshlang‘ich shartiga murojaat etamiz:

$$y(0)=3\Rightarrow(\frac{0^2}{2}+C)e^{-0^2}=3\Rightarrow C=3.$$

Demak, berilgan Koshi masalasining izlangan yagona yechimi

$$y=(\frac{x^2}{2}+3)e^{-x^2}$$

funksiyadan iborat bo‘ladi.

Bernulli tenglamasi. Bu tenglama

$$y'+P(x)y=Q(x)y^{\alpha} \tag{8}$$

ko‘rinishda bo‘lib, unda $\alpha\neq0$ va $\alpha\neq1$ shartni qanoatlantiruvchi ixtiyoriy haqiqiy sonni ifodalaydi. $\alpha=0$ yoki $\alpha=1$ holda (8)yuqorida ko‘rib o‘tilgan (3) chiziqli differensial tenglamaga aylanadi. Bernulli tenglamasini integrallash uchun uni dastlab y^{α} ifodaga bo‘lamiz. Natijada (8) tenglama

$$y^{-\alpha}y'+P(x)y^{1-\alpha}=Q(x)$$

ko‘rinishga keladi. Endi $z=y^{1-\alpha}$ belgilash kiritamiz. Bu holda, murakkab funksiya hosilasi formulasiga asosan, $z'=(1-\alpha)\,y^{-\alpha}y'$ bo‘lgani uchun oxirgi tenglama

$$\frac{1}{1-\alpha}z'+P(x)z=Q(x)\Rightarrow z'+(1-\alpha)P(x)z=(1-\alpha)Q(x) \tag{9}$$

ko‘rinishga, ya’ni chiziqli differensial tenglamaga keladi. Bu tenglamani yuqorida ko‘rsatilgan Bernulli usulida integrallab, $z=\varphi(x,C)$ umumiy yechimni topamiz. Unda, yuqoridagi belgilashga asosan, (8) Bernulli tenglamasining umumiy integrali $y^{1-\alpha}=\varphi(x,C)$ tenglik bilan aniqlanadi.

Misol sifatida, ushbu

$$y'+\frac{2}{x}y=2e^{x}\sqrt{y}$$

Bernulli tenglamasining umumiy integralini yuqorida ko‘rsatilgan usulda topamiz.

$$\frac{y'}{\sqrt{y}}+\frac{2}{x}\sqrt{y}=2e^{x}\Rightarrow(z=\sqrt{y}\,,\,z'=\frac{y'}{2\sqrt{y}})\Rightarrow z'+\frac{1}{x}z=e^{x}.$$

Hosil bo‘lgan chiziqli tenglamani integrallab,

$$z = e^x + \frac{C - e^x}{x} \Rightarrow \sqrt{y} = e^x + \frac{C - e^x}{x} \Rightarrow y = (e^x + \frac{C - e^x}{x})^2$$

natijani olamiz.

1-misol. $(y - x)ydx + x^2dy = 0$ tenglamaning umumiy integralini toping

Yechish. $y = xz$ deymiz. U holda $dy = xdz + zdx$ va tenglama $x^2(z-1)zdx + x^2(xdz + zdx) = 0$ yoki $z^2dx + xdz = 0$ ko‘rinishda bo‘ladi. O‘zgaruvchilarni ajratamiz:

$$\frac{dx}{x} + \frac{dz}{z^2} = 0 \text{ yoki } \int\frac{dx}{x} + \int\frac{dz}{z^2} = c, \qquad \ln x - \frac{1}{z} = c.$$

$z = \frac{y}{x}$ belgilash orqali y o‘zgaruvchiga qaytib, umumiy integralni topamiz: $\ln x - \frac{x}{y} = c.$

Bundan tashqari, $z = 0$ ham yechim bo‘ladi, bu yerda $y = 0$ ni hosil qilamiz.

2-misol. $y' + 2y = 4x$ tenglama berilgan.

Yechish: Berilgan tenglamada $P(x) = 2$; $Q(x) = 4x$ yechimni $y = U \cdot \vartheta$ ko‘rinishda izlaymiz: $\vartheta = e^{-\int 2dx} = e^{-2x}$; $u = \int \frac{4x}{e^{-2x}}dx + C = 4\int xe^{2x}dx + C$

$J = \int xe^{2x}dx$ integrallarni topishda bo‘laklab integrallash $\int ud\vartheta = U\vartheta - \int \vartheta du$ formulasidan foydalanamiz: $u = x$; $du = dx$ $J = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$; bundan $d\vartheta = e^{2x}dx$;

$$\vartheta = \frac{1}{2}e^{2x}$$

$u = (2x-1)e^{2x} + C$ topilgan u va ϑ larni qiymatlarini $y = U \cdot \vartheta$ ga qo‘yamiz:

$y = \left[(2x-1)e^{2x} + C\right] \cdot e^{-2x} = 2x - 1 + Ce^{-2x}$ Javob: $y = 2x - 1 + Ce^{-2x}$;

$y' + \frac{1-2x}{x^2}y = 1$

Yechish: Berilgan tenglamada $P(x) = \frac{1-2x}{x^2}$; $Q(x) = 1$; yecim $y = U \cdot \vartheta$ ko‘rinishda bo‘lsa, u va ϑ funksiyalarni topamiz.

$$\vartheta = e^{-\int \frac{1-2x}{x^2}dx} = e^{-\int\left(\frac{1}{x^2}-\frac{2}{x}\right)dx} = e^{\frac{1}{x}+2\ell n|x|} = e^{\frac{1}{2}} \cdot x^2;$$

$$u = \int \frac{1}{x^2 \cdot e^{-\frac{1}{x}}}dx + C = \int x^{-2} \cdot e^{-\frac{1}{x}}dx + C = \left|\int e^{-\frac{1}{x}}d\left(-\frac{1}{x}\right) = \int e^t dt = e^t = e^{-\frac{1}{x}}\right| = e^{-\frac{1}{x}} + C;$$

$$y = \left(e^{-\frac{1}{x}+C}\right) \cdot x^2 e^{\frac{1}{x}} = x^2 + Cx^2 e^{\frac{1}{x}} = x^2(1+C)e^{\frac{1}{x}}$$

Javob: $y = x^2 e\frac{1}{x} \cdot (1+C)$

3-misol. Ushbu Bernulli tenglamasining umumiy integralini toping:

$$\frac{dy}{dx} - \frac{3}{x} \cdot y = -x^3 y^2$$

Yechish: Ikkala tomonni y^2 ga bo‘lamiz:

$$\frac{1}{y^2} \cdot \frac{dy}{dx} - \frac{3}{x} \cdot \frac{1}{y} = -x^3$$

$z = \frac{1}{y}$ deb olamiz va ikkala tomonni x bo‘yichadifferensiallaymiz:

$$-\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

Buni e’tiborga olsak:

$$\frac{dz}{dx} + \frac{3}{x} \cdot z = x^3 \qquad (4)$$

Bu chiziqli tenglamani variatsiyalash usuli bilan integrallaymiz. Avval, bir jinsli tenglamani yechamiz:

$$\frac{dz}{dx} + \frac{3}{x}z = 0, \qquad \frac{dz}{z} + \frac{3}{x}dx = 0, \qquad \int\frac{dz}{z} + 3\int\frac{dx}{x} = \ln c$$

$$\ln z + 3\ln x = \ln c, \quad \ln zx^3 = \ln c, \quad z = \frac{c}{x^3} \ (**)$$

$c = c(x)$ deb, quyidagini hisoblaymiz:

$$\frac{dz}{dx} = \frac{dc(x)}{dx}\frac{1}{x^3} - \frac{3c(x)}{x^4} \qquad (4)$$

ga qo‘yamiz:

$$\frac{dc(x)}{dx}\frac{1}{x^3}-\frac{3c(x)}{x^4}+\frac{3}{4}\frac{c(x)}{x^3}=x^3$$

yoki

$dc(x)=x^6dx,\qquad c(x)=\frac{x^7}{7}+c_1\ \rightarrow(**)\quad z=\frac{x^4}{7}+\frac{c_1}{x^3}.$ z ni $\frac{1}{y}$ bilan almashtirib, ushbuni hosil qilamiz:

$$\frac{1}{y}=\frac{x^4}{7}+\frac{c_1}{x^3}\text{ yoki }\ y\left(\frac{x^7}{7}+c_1\right)=x^3.$$

To‘la differensial tenglama.

Hosilaga nisbatan yechilgan birinchi tartibli

$$\frac{dy}{dx}=f(x,y)$$

differensial tenglamani,

$$M(x,y)dx+N(x,y)dy=0$$

ko‘rinishga keltirish mumkin.Tenglamada $x\quad va\quad y$ o‘zgaruvchilar teng kuchli ravishda qatnashadilar.

Ta’rif: Agar tenglamaning chap tomoni biror $u(x,y)$ funksiyaning to‘liq differensialiga teng bo‘lsa, u holda quyidagi tenglamaga to‘liq differensilli tenglama deyiladi.

Ya`ni

$$M(x,y)dx+N(x,y)dy\equiv du=\frac{\partial u}{dx}dx+\frac{\partial u}{\partial y}dy \qquad (10)$$

Bundan

$$\frac{\partial u}{\partial x}=M(x,y),\qquad \frac{\partial u}{\partial y}=N(x,y) \qquad (11)$$

ga ega bo‘lamiz.

Agar berilgan tenglamaning biror yechimini (10) ga keltirib qo‘ysak $du=0$ ga ega bo‘lamiz.

Bundan $u(x,y)=c$ Bu esa, to‘liq differensial tenglamaning umumiy integralidir.

Faraz qilaylik $M(x,y)$ $ва$ $N(x,y)$ funksiyalari, mos ravishda y va x ga nisbatan uzluksiz xususiy hosilalarga ega bo‘lsin.

$$\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y \partial x} \qquad \frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x \partial y}$$

bundan,

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Bu tenglamaning chap tomoni biror funksiyani to‘liq differensiali bo‘lishligini zaruriy shartidir.

Bu yetarli shart ekanligini isbot qilamiz. Haqiqatdan ham tenglik bajarilganda shunday $u(x,y)$ funksiyani topish mumkinki, bu funksiya tengliklarni qanoatlantiradi.

Faraz qilaylik, u ning birinchisini qanoatlantirsin.

$$\frac{\partial u}{\partial x}=M(x,y)$$

Agar bunda u ni parametr deb olsak, uni

$$dy=M(x,y)dx$$

ko‘rinishda yozish mumkin.

$$u(x,y)=\int_{xo}^{x} M(x,y)dx+\varphi(y)$$

Buning har ikkala tomonini u ga nisbatan differensiallaymiz.

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\int_{xo}^{x} M(x,y)dx+\varphi'(y)=\int_{xo}^{x}\frac{\partial M}{\partial y}dx+\varphi'(y)=\int_{xo}^{x}\frac{\partial N}{\partial x}dx+\varphi'(y)=N(x,y)\Big|_{xo}^{x}+\varphi'(y)=$$

$$=N(x,y)-N(x_0,y)+\varphi'(y)$$

ni e`tiborga olsak, keyingi tenglikdan

$$\varphi'(y)=N(x_0,y)$$

ga ega bo‘lamiz. Bundan

$$\varphi(y)=\int_{y_0}^{y} N(x_0,y)dy$$

$$\int_{x0}^{x} M(x,y)dx+\int_{y0}^{y} N(x_0,y)dy=c$$

Bu ko‘rsatadiki, bajarilganda ni chap tomoni to‘liq differensial tenglama bo‘ladi. to‘liq differensialli tenglamaning umumiy integralini topish formulasi.

Misol:

$$(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$$

$$M(x,y)=3x^2+6xy^2 \qquad N(x,y)=6x^2y+4y^3$$

$$\frac{\partial M}{\partial y}=12xy \qquad \frac{\partial N}{\partial x}=12xy \quad demak, \frac{\partial M}{\partial y}\equiv\frac{\partial N}{\partial x}$$

Topshiriqlar.

Quyidagi differensial tenglamalarni yeching.

1. $y'+2xy=2x^3y^3$ **2.** $y^{n-1}(ay+y)=x$

3. $xy'+y=y^2\ell nx;$ **4.** $y'+\frac{2y}{x}=\frac{2\sqrt{y}}{\cos^2 x}$

5. $y'+2xy=xe^{-x^2};$ **6.** $(1+x^2)y'-2xy=(1+x^2)^2;$

7. $y'+ay=e^{mx}$ **8.** $y'=\frac{1}{2x-y^2};$

9. $x(y'-y)=(1+x^2)e^x;$ **10.** $y'+\frac{y}{x+1}+y^2=0;$

11. $xdx=\left(\frac{x^2}{y}-y^2\right)dy;$ **12.** $y'-ytgx+y^2\cos x=0;$

13. $xy'-4y-x^2\sqrt{y}=0;$ **14.** $y'=\frac{y^2}{x^2}-2$

15. $xdy-ydx=ydy$ **16.** $y^2+x^2y'=xyy'$

17. $y^1=\frac{x+y}{x-y}$ **18.** $(3y^2+3xy+x^2)dx=(x^2+2xy)dy$

19. $y'=\frac{y}{2y\ell ny+y-x}$ **20.** $(y-x)ydx+x^2dy=0$

4-MAVZU. YUQORI TARTIBLI DIFFERENSIAL TENGLAMALAR. TARTIBI PASAYTIRILADIGAN DIFFERENSIAL TENGLAMALAR

Ushbu $\frac{d^2y}{dx^2} = f(x, \frac{dy}{dx})$ (1) ko‘rinishdagi tenglama noma’lum y funksiyani oshkor holda o‘z ichiga olmaydi. Bunday tenglamani yechish uchun $\frac{dy}{dx} = p$ belgilash kiritiladi. Bunda $\frac{d^2y}{dx^2} = \frac{dp}{dx}$ ($y'' = p'$). Bu ifodalarni (1) tenglamaga qo‘yib, x ning noma’lum p funksiyaga nisbatan birinchi tartibli $\frac{dp}{dx} = f(x, p)$ tenglamani hosil qilamiz. Bu tenglamani integrallab, uning $p = p(x, C_1)$ - umumiy yechimini topamiz, $\frac{dy}{dx} = p$ tenglikdan (1) tenglamaning $y = \int p(x, C_1)dx + C_2$ - umumiy integralini topamiz.

1-misol. Zanjir chiziqning $\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ differensial tenglamasini qaraymiz.

Yechish: $\frac{dy}{dx} = p$ deb olamiz; u holda $\frac{d^2y}{dx^2} = \frac{dp}{dx}$ bo‘lib, $\frac{dp}{dx} = \frac{1}{a}\sqrt{1 + p^2}$ differensial tenglama hosil bo‘ladi. O‘zgaruvchilarni ajratsak, $\frac{dp}{\sqrt{1 + p^2}} = \frac{dx}{a}$; bundan

$$\ln\left(p + \sqrt{1 + p^2}\right) = \frac{x}{a} + C_2;\ p = sh\left(\frac{x}{a} + C_1\right);\ \frac{dy}{dx} = sh\left(\frac{x}{a} + C_1\right) + C_2$$

x erkli o‘zgaruvchini oshkor holda o‘z ichiga olmagan $\frac{d^2y}{dx^2} = f\left(y, \frac{dy}{dx}\right)$ (2) ko‘rinishdagi tenglama. Bu tenglamani yechish uchun $\frac{dy}{dx} = p$ (3) deb olamiz. Ammo endi p x ning funksiyasi emas, balki y ning funksiyasi bo‘ladi. Bu holda

$\frac{d^2y}{dx^2} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} \cdot p$ yoki $y' = p$ hamda $y'' = pp'$ bo‘ladi. Yuqoridagi almashtirishlarni (2) tenglamaga qo‘yib, $p\frac{dp}{dy} = f(y, p)$ (4) tenglamani hosil qilamiz.

Buni integrallab, p ni y va ixtiyoriy C_1 o‘zgarmas miqdorning funksiyasi kabi aniqlaymiz: $p = p(y, C_1)$ (3) tenglikdan foydalanib $\frac{dy}{dx} = p(y, C_1)$ differensial tenglamaga ega bo‘lamiz. O‘zgaruvchilarni ajratib, $\frac{dy}{p(y, C_1)} = dx$ tenglamaga ega bo‘lamiz. Oxirgi tenglamani integrallab, $\Phi(x, y, C_1, C_2) = 0$ umumiy integralini topamiz.

2-misol $3y'' = y^{-\frac{5}{3}}$ tenglamaning umumiy integralini toping.

Yechish: p ni y ning funksiyasi ekanini bilgan holda $p = \frac{dy}{dx}$ deb olamiz. Bu holda $y'' = p\frac{dp}{dx}$ bo‘ladi. $3p\frac{dp}{dy} = y^{-\frac{5}{3}}$ buni integrallab

$$p^2 = C_1 - y^{-\frac{2}{3}} \text{ yoki } p = \pm\sqrt{C_1 - y^{-\frac{2}{3}}} \cdot p = \frac{dy}{dx}$$

ni hisoblab olsak,

$$\pm\frac{dy}{\sqrt{C_1 - y^{-\frac{2}{3}}}} = dx \text{ yoki } \frac{y^{\frac{1}{3}}dy}{\pm\sqrt{C_1 y^{\frac{2}{3}} - 1}} = dx$$

tenglamani hosil qilamiz. Bundan: $x + C_2 = \pm\int\frac{y^{\frac{1}{3}}dy}{\sqrt{C_1 y^{\frac{2}{3}} - 1}}$ - integralni hisoblash uchun

$C_1 y^{\frac{2}{3}} - 1 = t^2$ almashtirishni bajarib, $y^{\frac{1}{3}} = \left(t^2 + 1\right)^{\frac{1}{2}}\frac{1}{C_1^{\frac{1}{2}}}$; $dy = 3t\left(t^2 + 1\right)^{\frac{1}{2}}\frac{1}{C^{\frac{3}{2}}}dt$ ifodaga

ega bo‘lamiz. $\int\frac{y^{\frac{1}{3}}dy}{\sqrt{C_1 y^{\frac{2}{3}} - 1}} = \frac{1}{C_1^2}\int\frac{3t\left(t^2 + 1\right)}{t}dt = \frac{3}{C_1^2}\left(\frac{t^3}{3} + t\right) = \frac{1}{C_1^2}\sqrt{C_1 y^{\frac{2}{3}} - 1}\cdot\left(C_1 y^{\frac{2}{3}} + 2\right)$

Javob: $x + C_2 = \pm\frac{1}{C_1^2}\sqrt{C_1 y^{\frac{2}{3}} - 1}\left(C_1 y^{\frac{2}{3}} + 2\right)$

(1) va (2) ko‘rinishdagi differensial tenglamalarga doir nomerli misollarni yechilishidan namunalar keltiramiz.

Topshiriqlar:

1. $y'' = arctgx$**;** **2.** $xy'' = y'$

3. $y'' = \frac{y'}{x} + x$ **4.** $2xy'y'' = (y')^2 + 1$

5. $1 + (y')^2 = 2yy''$ **6.** $y'' + \frac{2}{1-y}(y')^2 = 0$

7. $yy'' = (y')^2$ **8.** $y(1 - \ln y)y'' + (1 + \ln y)(y')^2 = 0$

9. $yy'' - yy' \ln y = (y')^2$ **10.** $(x + a)y'' + x(y')^2 = y'$

Topshiriqlar:

1. $y'' = arctgx$; **2.** $xy'' = y'$

3. $y'' = \frac{y'}{x} + x$ **4.** $2xy'y'' = (y')^2 + 1$

5. $1 + (y')^2 = 2yy''$ **6.** $y'' + \frac{2}{1-y}(y')^2 = 0$

7. $yy'' = (y')^2$ **8.** $y(1 - \ln y)y'' + (1 + \ln y)(y')^2 = 0$

9. $yy'' - yy' \ln y = (y')^2$ **10.** $(x + a)y'' + x(y')^2 = y'$

11. $y'' = x + \sin x$ **12.** $y'' = \ln x$

13. $y'' = y' + x$ **14.** $(1 + x^2)y'' + (y')^2 + 1 = 0$

15. $(y'')^2 = y'$ **16.** $y'' - 2ctgx \cdot y' = \sin^2 x$

17. $(y')^2 + 2yy'' = 0$ **18.** $y'' = \frac{1}{4\sqrt{y}}$

19. $yy'' + (y')^2 = 1$ **20.** $2yy'' - 3(y')^2 = 4y^2$

21. $y'' = 2yy'$ **22.** $yy'' - (y')^2 = y^2 \cdot y'$

5-MAVZU. O‘ZGARMAS KOEFFITSIENTLI YUQORI TARTIBLI CHIZIQLI BIR JINSLI DIFFERENSIAL TENGLAMALAR. O‘ZGARMAS KOEFFITSIENTLI YUQORI TARTIBLI CHIZIQLI BIR JINSLI BO‘LMAGAN, O‘NG TOMONI MAXSUS KO‘RISHISHGA EGA BO‘LGAN DIFFERENSIAL TENGLAMALAR

Ikkinchi tartibli bir jinsli chiziqli tenglama

$$y'' + py' + qy = 0 \qquad (1)$$

berilgan bo‘lsin, bunda p va q – o‘zgarmas haqiqiy sonlar. Yuqorida tenglamaning umumiy integralini topish uchun uning ikkita chiziqli erkli xususiy yechimini topish yetarlidir.

Xususiy yechimlarini $y = e^{kx}$ (2) bunda k=const ko‘rinishda izlaymiz, bu holda

$$y' = ke^{kx};\ \ y'' = k^2 e^{kx}$$

Hosilalarning bu ifodalarini (1) tenglamaga qo‘ysak, u $e^{kx}\left(k^2 + px + q\right) = 0$ bundan $e^{kx} \neq 0$ bo‘lgani uchun $k^2 + pk + q = 0$ (3) bo‘lib, (3) tenglamani qanoatlantirsa, u holda e^{kx} (1) tenglamaning yechimi bo‘ladi. (3) tenglama (1) tenglamaning *xarakteristik tenglamasi* deyiladi.

Xarakteristik tenglama 2 ta ildizi bo‘lgan kvadrat tenglamadir, bu ildizlarni k_1 va k_2 bilan belgilaymiz . Bunda

$$k_1 = -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q};\ \ k_2 = -\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

Quyidagi hollar bo‘lishi mumkin :

1. k_1 va k_2 - haqiqiy va bir – biriga teng bo‘lmagan sonlar $\left(k_1 \neq k_2\right)$.
2. k_1 va k_2 – haqiqiy va bir – biriga teng sonlar $\left(k_1 = k_2\right)$.
3. k_1 va k_2 – kompleks sonlar.

Bu hollarni har birini alohida ko‘ramiz :

1. Xarakteristik tenglamaning ildizlari haqiqiy va har xil $(k_1 \neq k_2)$ bo‘lgan hol.

Bu holda $y_1 = e^{k_1 x}$; $y_2 = e^{k_2 x}$ funksiyalar xususiy yechimlari bo‘ladi.

Bu yechimlar $\frac{y_2}{y_1} = \frac{e^{k_2 x}}{e^{k_1 x}} = e^{(k_2 - k_1)x} \neq const.$

bo‘lgani uchun chiziqli erkli bo‘ladi.

Demak, umumiy integral $y = c_1 e^{k_1 x} + c_2 e^{k_2 x}$ bo‘ladi.

2. Xarakteristik tenglamaning ildizlari haqiqiy va teng bo‘lgan hol.

Bu holda $(k_1 = k_2)$

Demak, $y_1 = e^{k_1 x}$ - bitta xususiy yechim topiladi. Birinchi xususiy yechim topiladi. Birinchi xususiy yechim bilan chiziqli erkli bo‘lgan 2 – xususiy yechimni topish kerak. ($e^{k_2 x}$ funksiya $e^{k_1 x}$ ga aynan teng bo‘lgani uchun uni 2 – xususiy yechim sifatida olish mumkin emas.) 2 – xususiy yechimni $y_2 = U(x)e^{k_1 x}$ ko‘rinishda izlaymiz, bunda U(x) – aniqlanishi kerak bo‘lgan noma’lum funksiya. Differensiallab, quyidagilarni topamiz:

$$y_2' = U'e^{k_1 x} + k_1 U e^{k_1 x} = e^{k_1 x}(u' + k_1 U)$$

$$y_2'' = U''e^{k_1 x} + k_1 U e^{k_1 x} + k_1{}^2 U e^{k_1 x} = e^{k_1 x}(U'' + 2k_1 U' + k_1^2 U)$$

Hosilalarning bu ifodalarni (1) tenglamaga qo‘yib,

$$e^{k_1 x}\left[U'' + (2k_1 + p)U' + (k_1^2 + pk_1 + .q)U\right] = 0$$

tenglamani hosil qilamiz. Lekin k_1 xarakteristik tenglamaning karrali ildizi bo‘lgani uchun bu holda $k_1^2 + pk_1 + q = 0$

Bundan tashqari $k_1 = k_2 = -\frac{p}{2}$ yoki $2k_1 = -p;\ 2k_1 + p = 0$

Demak, U(x) ni topish uchun $e^{k_1 x}U'' = 0$ yoki U''=0 tenglamani yechish kerak. Bu tenglamani 2 marta integrallab U=Ax+B ekanini topamiz. Xususiy holda

A=1; B=0 deb olish mumkin, bu holda U=x bo‘ladi. Shunday qilib, 2 – xususiy yechim sifatida $y_2 = xe^{k_1 x}$ funksiyani olish mumkin. $\frac{y_2}{y} = x \neq const$ bo‘lgani bu yechim bilan 1 – xususiy yechim chiziqli erklidir. Shuning uchun

$$y = c_1 e^{k_1 x} + c_2 x e^{k_1 x} = e^{k_1 x}\left(c_1 + c_2 x\right)$$

funksiya umumiy integral bo‘ladi.

3. Xarakteristik tenglamaning ildizlari kompleks sonlar bo‘lgan hol. kompleks ildizlar juft – juft qo‘shma kompleks sonlar bo‘lgani uchun ularni

$$k_1 = \alpha + i\beta \quad k_2 = \alpha - i\beta$$

$$\alpha = -\frac{p}{2}, \quad \beta = \sqrt{q - \frac{p^2}{4}};$$

Xususiy yechimlarni,

$$y_1 = e^{(\alpha + i\beta)x} \; ; \; y_2 = e^{(\alpha - i\beta)x} \tag{4}$$

shaklida yozish mumkin. Bular haqiqiy argumentning kompleks funksiyalari bo‘lib, (1) differensial tenglamani qanoatlantiradi. Agar haqiqiy argumentning biror kompleks funksiyasi

$$y = U(x) + i\vartheta(x) \tag{5}$$

(1) tenglamani qanoatlantirsa, bu tenglamani *U(x)* va $\vartheta(x)$ funksiyalar ham qanoatlantiradi.

Haqiqatan (5) ifodani (1) tenglamaga qo‘ysak,

$$\left[U(x) + i\vartheta(x)\right]'' + p\left[U(x) + i\vartheta(x)\right]' + q\left[U(x) + i\vartheta(x)\right] = 0$$

yoki

$$\left(U'' + pU' + qU\right) + i\left(\vartheta'' + p\vartheta' + q\vartheta\right) = 0$$

tenglamani hosil qilamiz. Bundan

$$U'' + pU' + qU = 0 \; ; \; \vartheta'' + p\vartheta' + q\vartheta = 0$$

Demak, U(x) va $\vartheta(x)$ funksiyalar tenglamaning yechimi ekanini isbot etdik.

$$y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x}\cos\beta x + ie^{\alpha x}\sin\beta x$$

$$y_2 = e^{\alpha x}\cos\beta x - ie^{\alpha x}\sin\beta x$$

Isbot qilinganiga ko‘ra,

$$\tilde{y}_1 = e^{\alpha x}\cos\beta x,\ \tilde{y}_2 = e^{\alpha x}\sin\beta x$$

haqiqiy funksiyalar ham (1) tenglamaning xususiy yechimlari bo‘ladi.

$$\frac{\tilde{y}_1}{\tilde{y}_2} = \frac{e^{\alpha x}\cos\beta x}{e^{\alpha x}\sin\beta x} = ctg\beta x \neq const$$

bo‘lgani uchun $\tilde{y}_1$ va $\tilde{y}_2$ funksiyalar chiziqli erklidir. Demak, xarakteristik tenglama kompleks ildizlarga ega bo‘lganda (1) tenglamaning umumiy yechimi

$$y = c_1\tilde{y}_1 + c_2\tilde{y}_2 = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x$$

yoki

$$y = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)$$

ko‘rinishda bo‘ladi, bunda c_1 va c_2 – ixtiyoriy o‘zgarmas miqdorlar.

Agar (1) da *P=0* bo‘lsa, tenglama $y'' + qy = 0$ ko‘rinishda bo‘lib, xarakteristik tenglama $k^2+q=0$ $q>0$ ning ildizlari $k_{1,2} = \pm i\sqrt{q} = \pm i\beta$, $(\alpha = 0)$ -bo‘ladi.

Umumiy yechim esa , $y = c_1\cos\beta x + c_2\sin\beta x$

1 - misol $y'' + y' - 2y = 0$ tenglamaning umumiy integrali topilsin.

<u>Yechish</u>: berilgan tenglamaning xarakteristikasini tuzamiz. U $k^2 + k - 2 = 0$ ko‘rinishda bo‘ladi.

Xarakteristik tenglamaning ildizlari: $k_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2} \Rightarrow k_1 = 1; k_2 = -2$

Demak, umumiy integral

$$y = c_1e^{x} + c_2e^{-2x}$$

2- misol: $y'' + 4y' + 4y = 0$ tenglamaning umumiy integralini toping.

<u>Yechish</u>: $k^2+4k+4=0$ xarakteristik tenglamani tuzamiz. Bu tenglamaning ildizlari $(k+2)^2 = 0$ $k_1 = k_2 = 2$ Umumiy integral:

$$y = c_1e^{2x} + c_2xe^{2x}$$

3- misol . $y''+2y'+5y=0$ berilgan. y=0 ; y` = 1 boshlang'ich shartlarni qanoatlantiruvchi umumiy integrali va xususiy yechimi topilsin.

Yechish: $k^2+2k+5=0$ - xarakteristik tenglamani tuzamiz. Uning ildizlari $k_1=-1+2i,\quad k_2=-1-2i,$ ga teng.

Demak, umumiy integral

$$y=e^{-x}\left(c_1\cos\beta x+c_2\sin\beta x\right)$$

bo‘ladi.

Xususiy yechimni topamiz, ya'ni c_1 va c_2 qiymatlarni topamiz.

$$0=e^{-0}\left(c_1\cos 2\cdot 0+c_2\sin 2\cdot 0\right)$$

bundan $c_1=0$; $y'=e^{-x}2c_2\cos 2x-e^{-x}c_2\sin 2x$ dan

$$1=e^{-0}\cdot 2c_2\cos 2\cdot 0-e^{-0}c_2\sin 2\cdot 0\Rightarrow 1=2c_2\Rightarrow c_2=\frac{1}{2}$$

Xususiy yechim: $y=\frac{1}{2}e^{-x}\sin 2x$.

4. $y''+y'-2y=0$

Yechish: $k^2+k-2=0$ xarakteristik tenglamani tuzib, uni $k_1=-2$, $k_2=1$ yechimlarini topamiz.

Umumiy integral: $y=c_1e^{-2x}+c_2e^{x}$;

Ushbu $y''+py'+qy=f(x)$ (6) tenglama berilgan bo‘lsin, bunda p va q - haqiqiy sonlar. Bunday tenglamalarni yechish uchun avvalo $y''+py'+qy=0$ bir jinsli qismini yechimi topiladi, so‘ngra o‘ng tomonini ko‘rinishiga ko‘ra xususiy yechim topiladi. Umumiy yechim esa bir jinsli qismini $\bar{y}$ yechimi va y^* xususiy yechimini yig'indisi shaklida yozamiz: $y=\bar{y}+y^*$;

Bir jinsli qismini yechishni yuqorida ko‘rdik. Endi o‘ng tomonini ko‘rinishi quyidagi hollardan iborat bo‘lganda xususiy yechimni topishni ko‘rib chiqamiz.

1 – hol (6) tenglamaning o‘ng tomoni ko‘rsatkichli funksiya bilan ko‘phad ko‘paytmasidan iborat, ya'ni

$$f(x) = P_n(x)e^{\alpha x} \tag{7}$$

ko‘rinishda bo‘lsin, bunda $P_n(x)$ - n – darajali ko‘phad. U vaqtda quyidagi xususiy holler bo‘lishi mumkin:

a) α soni $k^2 + pk + q = 0$ xarakteristik tenglamaning ildizi bo‘lmagan hol.

Bu holda xususiy yechimni

$$y^* = \left(A_o x^n + A_1 x^{n-1} + \ldots + A_n\right)e^{\alpha x} = Q_n(x)e^{\alpha x} \tag{8}$$

ko‘rinishda izlash kerak. Haqiqatan, y^* ni tenglamaga qo‘yib va barcha hadlarni $e^{\alpha x}$ ko‘paytuvchiga qisqartirib,

$$Q_n''(x) + (2\alpha + p)Q_n'(x) + \left(\alpha^2 + p\alpha + q\right)Q_n(x) = P_n(x)$$

tenglikka ega bo‘lamiz, bunda $Q_n(x)$ - darajali ko‘phad, $Q'(n)$ (n-1) – darajali ko‘phad, $Q''(n)$ esa (n-2) – darajali ko‘phad. Shunday qilib, tenglik ishorasining o‘ng va chap tomonlarida n – darajali ko‘phadlar turibdi. Bir xil darajali x lar oldidagi koeffitsientlarini bir-biriga tenglab, noma’lum A_o, A_1…..A_n koeffitsientlarni topish uchun tenglamalarningsoni (n+1) ta bo‘lgan sistema hosil qilamiz.

b) α - xarakteristik tenglamaning oddiy (bir karrali) ildizi bo‘lgan holda xususiy yechimni $y* = xQ_n(x)e^{\alpha x}$ ko‘rinishda izlaymiz.

c) α son xarakteristik tenglamaning 2 karrali ildizi bo‘lgan hol. Bu holda $Q_n(x)e^{\alpha x}$ funksiyani differensial tenglamaga qo‘yish natijasida ko‘phadning darajasi 2 birlik pasayadi. Haqiqatan ham, agar α - xarakteristik tenglamaning ildizi bo‘lsa, u holda $\alpha^2 + p\alpha + q = 0$, bundan tashqari, α ikki karrali ildiz bo‘lgani uchun $2\alpha = -p \Rightarrow 2\alpha + p = 0$.

Demak, (8) tenglikning chap tomonida $Q''(x)$ ya’ni (n-2) – darajali ko‘phad qoladi. O‘rniga qo‘yish natijasida n – darajali ko‘phad tashkil qilish uchun xususiy yechimni $e^{\alpha x}$ bilan (n+2) – darajali ko‘phadning ko‘paytmasi shaklida izlash kerak.

Demak, α xarakteristik tenglamaning 2 karrali ildizi bo‘lgan holda xususiy

yechimni $y^* = x^2 Q_n(x)e^{\alpha x}$ ko‘rinishda izlaymiz.

Misol: $y'' - 7y' + 6y = (x-2)e^x$

<u>Yechish</u>: Bu tenglamaning o‘ng tomoni $P_1(x)e^{1\cdot x}$ ko‘rinishda bo‘lib, daraja ko‘rsatkichdagi 1 koeffitsient xarakteristik ko‘phadning oddiy ildizi. Demak, xususiy yechimni $y^* = xQ_1(x)e^x$ yoki $y^* = x(Ax+B)e^x$ ko‘rinishda izlaymiz; Bu ifodani tenglamaga qo‘yib,

$$\left[\left(Ax^2 + Bx\right) + 4Ax + 2B + 2A - 7(Ax^2 + Bx) - 7(2Ax + B) + 6(Ax^2 + Bx)\right]e^x = (x-2)e^x$$

yoki,

$$\left(-10Ax - 5B + 2A\right)e^x = (x-2)e^x \text{ ni}$$

hosil qilamiz. Bundan esa,

$$\begin{cases} -10A = 1 \\ 2A - 5B = -2 \end{cases} \quad A = -\frac{1}{10}\ ; \quad B = \frac{9}{25}.$$

Demak, $y^* = x\left(-\frac{1}{10}x + \frac{9}{25}\right)e^x$; $y = c_1 e^{6x} + c_2 e^x + x\left(-\frac{1}{10}x + \frac{9}{25}\right)e^x$;

2 – hol. O‘ng tomon

$$f(x) = P(x)e^{\alpha x}\cos\beta x + Q(x)e^{\alpha x}\sin\beta x \qquad (9)$$

ko‘rinishda bo‘lsin. Bunda P(x) va Q(x) ko‘phadlar. U holda xususiy yechimning ko‘rinishi quyidagicha aniqlanadi.

a) $\alpha + i\beta$ son xarakteristik tenglamaning ildizi bo‘lmasa, u holda (6) tenglamaning xususiy yechimini

$$y^* = U(x)e^{\alpha x}\cos\beta x + \vartheta(x)e^{\alpha x}\sin\beta x \qquad (10)$$

ko‘rinishda izlash kerak, bunda $U(x)$ va $\vartheta(x)$ - darajasi P(x) va Q(x) ko‘phadlarning eng yuqori darajasiga teng bo‘lgan ko‘phadlardir;

b) $\alpha + i\beta$ soni xarakteristik tenglamaning ildizi bo‘lsa, u holda xususiy yechimni

$$y^* = x\left[U(x)e^{\alpha x}\cos\beta x + \vartheta(x)e^{\alpha x}\sin\beta x\right] \qquad (11)$$

ko‘rinishda izlaymiz. Agar P(x) yoki Q(x) ko‘phadlarning birortasi 0 ga teng

bo‘lsa, ya’ni o‘ng tomoni $P(x)e^{\alpha x}\cos\beta x$ yoki $Q(x)e^{\alpha x}\sin\beta x$ ko‘rinishda bo‘lgan vaqtda ham y^* ni yuqoridagi ko‘rinishi saqlanadi.

2 – tartibli chiziqli tenglamaning o‘ng tomoni

$$f(x) = M\cos\beta x + N\sin\beta x \qquad (8)$$

ko‘rinishda bo‘lsin.

a) Agar β_i soni xarakteristik tenglamaning ildizi bo‘lsa, xususiy yechimni

$$y^* = x\left(A\cos\beta x + B\sin\beta x\right)$$

ko‘rinishda izlash kerak.

Misol: $y'' + 4y' = \cos 2x$

Yechish: $k^2+4=0$ xarakteristik tenglamaning ildizlari $k_1=2i$ $k_2=-2i$ bo‘ladi.

Bir jinsli tenglamaning umumiy yechimi

$$\bar{y} = c_1\cos 2x + c_2\sin 2x$$

Bir jinslimas tenglamaning xususiy yechimini

$$y^* = x\left(A\cos 2x + B\sin 2x\right)$$

ko‘rinishda izlaymiz. Bu holda

$$y^{*\prime} = 2x\left(-A\sin 2x - B\cos 2x\right) + 4\left(-A\sin 2x + B\sin 2x\right)$$

$$y^{*\prime\prime} = -4x\left(-A\cos 2x - B\sin 2x\right) + 4\left(-A\sin 2x + B\cos 2x\right)$$

Bu ifodalarni berilgan tenglamaga qo‘yib va cos2x hamda sin2x oldidagi koeffitsientlarni bir-biriga tenglab, A va B ni aniqlash uchun 4B=1; 4A=0 tenglamalar sistemasini hosil qilamiz, bunda A=0 $B = \frac{1}{4}$.

Demak, berilgan tenglamaning umumiy integrali :

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}x\sin 2x.$$

Topshiriqlar.

1. $2y'' + y' - y = 2e^x$

2. $y'' - 7y' + 6 = \sin x$

3. $y'' - 6y' + 9y = 2x^2 - x + 3$

4. $2y'' + 5y' = f(x)$

5. $f(x) = \cos^2 x$

6. $f(x) = 0{,}1e^{-2.5x} - 25\sin 2{,}5x$

7. $f(x) = 25x\sin x$

8. $f(x) = 100\,xe^{-x}\cos x.$

9. $y'' - 9y = 0$

10. $y'' - y' - 2y = 0$

11. $4y'' - 8y' + 5y = 0$

12. $4\dfrac{d^2x}{dt^2} - 20\dfrac{dx}{dt} + 25x = 0$

13. $y'' - 4y' + 3y = 0$; $y = 0$; $y' = 10$

14. $4y'' + 4y' + y = 0$; y=2, $y' = 0$

15. $3y'' - 2y' - 8y = 0$

16. $y'' + 6y' + 13y = 0$

17. $y'' + 4y' + 29y = 0$ y=0, $y' = 15$

18. $2y'' + y' + 2\sin 15^o \cos 15^o y = 0$

19. $y'' - 4y' = 0$

20. $y'' - 2y' + y = 0$

6-MAVZU. MUSBAT HADLI SONLI QATORLAR. QATOR YIG‘INDISI. QATOR YAQINLASHISHINING ZARURIY SHARTLARI. MUSBAT HADLI SONLI QATORLARNI TAQQOSLASH

$$a_1 + a_2 + ... + a_n + \ldots = \sum_{n=1}^{\infty} a_n \qquad (1)$$

$S_n = a_1 + a_2 + ... + a_n$ - qismiy yig‘indi, $S = \lim_{n\to\infty} S_n$ - (1) ning yig‘indisi (2). Agar (2) ning limiti 0 ga teng bo‘lsa (1) qator yaqinlashuvchi bo‘ladi, bu qator yaqinlashishining zaruriy shartidir.

Agar $S = \pm\infty$ (3) yoki S mavjud bo‘lmasa, (1) – uzoqlashuvchi

$$a_{n+1} + a_{n+2} + ... = \sum_{k=n+1}^{\infty} a_k \qquad (4)$$

Teorema. (1) qator yaqinlashuvchi bo‘lsa, (4) qator ham yaqinlashuvchi bo‘ladi va aksincha.Umumlashgan garmonik qator:

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + ... + \frac{1}{n^k} + ... = \sum_{n=1}^{\infty} \frac{1}{n^k} \qquad (5)$$

(5) da k=1 bo‘lsa, garmonik qator hosil bo‘ladi:

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ... = \sum_{n=1}^{\infty} \frac{1}{n} \qquad (6)$$

k>1 da (5) qator yaqinlashuvchi , $k \le 1$ bo‘lganda uzoqlashuvchi bo‘ladi.

Geometrik qator

$$a + a \cdot q + a \cdot q^2 + ... + a \cdot q^n + ... \qquad (7)$$

$|q|<1$ da (7) geometrik qator yaqinlashuvchi , $|q| \ge 1$ bo‘lganda uzoqlashuvchi bo‘ladi.

1-misol. Umumiy hadi $a_n = \frac{1}{(2n-1)\cdot(2n+1)}$ bo‘lgan qator yig‘indisini toping.

Yechilishi: n ga ketma-ket 1,2,3,… qiymatlar berib, ushbu

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ... + \frac{1}{(2n-1)\cdot(2n+1)} + ... = \sum_{n=1}^{\infty} \frac{1}{(2n-1)\cdot(2n+1)}$$

qatorni hosil qilamiz. Qatorning S_n xususiy yig‘indisi

$$S_n = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \ldots + \frac{1}{(2n-1)\cdot(2n+1)}$$

ga teng S_n xususiy yig‘indisini soddaroq ko‘rinishga keltirish uchun umumiy hadi $a_n = \frac{1}{(2n-1)\cdot(2n+1)}$ ni aniqmas koeffitsientlar usuli bo‘yicha sodda kasrlarga ajratamiz:

$$\frac{1}{(2n-1)\cdot(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

yoki, 1=(2A+2B)n +(A-B) $\Rightarrow$ 2A+2B =0 1=A-B $\Rightarrow$ A=$\frac{1}{2}$, $B = -\frac{1}{2}$.

U holda,

$$\frac{1}{(2n-1)\cdot(2n+1)} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$

Bu formulani har bir hadga qo‘llasak,

$$S_n = (\frac{1}{2} - \frac{1}{2\cdot 3}) + (\frac{1}{2\cdot 3} - \frac{1}{2\cdot 5}) + (\frac{1}{2\cdot 5} - \frac{1}{2\cdot 7}) + \ldots + (\frac{1}{2(2n-3)} - \frac{1}{2(2n-1)}) + (\frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}) =$$
$$= \frac{1}{2} + \frac{1}{2(2n+1)}$$

Qator yig‘indisi S ni (2) formuladan topamiz:

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}(\frac{1}{2} - \frac{1}{2(2n+1)}) = \frac{1}{2} - \frac{1}{2}\cdot\lim_{n\to\infty}\frac{1}{2n+1} = \frac{1}{2}$$

Demak, berilgan qator yaqinlashuvchi bo‘lib, uning yig‘indisi S=$\frac{1}{2}$ ga teng bo‘ladi.

2– misol. Koshi kriteriyasidan foydalanib,

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

garmonik qatorning uzoqlashuvchi ekanligini ko‘rsating.

Yechilishi: Koshi kriteriyasiga asosan $\forall E = \frac{1}{4}$, $\exists N$, $n > N$ va $p = n$ lar uchun $|S_{2n} - S_n| < \frac{1}{4} = E$ bo‘lishi kerak. Lekin,

$$|S_{2n} - S_n| = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > n\cdot\frac{1}{2n} = \frac{1}{2} > E$$

Bundan esa garmonik qatorning uzoqlashuvchiligi kelib chiqadi.

3-misol. $\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+\ldots$ qator uchun yaqinlashishning zaruriy sharti bajariladimi?

Yechilishi. Qatorning umumiy hadi $a_n=\frac{2n}{2n+1}$ ko‘rinishda bo‘ladi. U holda

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{2n}{2n+1}=\lim_{n\to\infty}\frac{2}{2+\frac{1}{n}}=\frac{2}{2+\lim\frac{1}{n}}=\frac{2}{2+0}=1\neq 0$$

Quyidagi qatorlarning yaqinlashish yoki uzoqlashishini tekshiring.

4-misol. $\sum_{n=1}^{\infty}\frac{1}{n\cdot 3^{n-1}}$.

Yechish. Berilgan qatorning $a_n=\frac{1}{n\cdot 3^{n-1}}$ umumiy hadi yaqinlashuvchi bo‘lgan quyidagi $\sum_{n=1}^{\infty}\frac{1}{3^{n-1}}$ geometrik $\left(q=\frac{1}{3}<1\right)$ qatorning umumiy hadidan kichik, ya’ni $a_n=\frac{1}{n\cdot 3^{n-1}}\le\frac{1}{3^{n-1}}=b_n$ solishtirish alomatiga asosan, berilgan qator yaqinlashuvchi bo‘ladi.

5-misol. $\sum_{n=1}^{\infty}\frac{1}{2n-1}$.

Yechish. Umumiy hadi $a_n=\frac{1}{2n-1}$ bo‘lgan berilgan qatorni umumiy hadi $b_n=\frac{1}{n}$ bo‘lgan uzoqlashuvchi $\sum_{n=1}^{\infty}\frac{1}{n}$ garmonik qator bilan solishtiramiz.

Ikkinchi solishtirish alomatiga asosan

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1}{2n-1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{2n-1}=\frac{1}{2}\neq 0$$

bo‘lgani uchun berilgan qator ham uzoqlashuvchi bo‘ladi.

6-misol. $\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+\ldots$ qator yaqinlashishning zaruriy shart bajarilishini tekshiring.

Yechish. Berilgan qatorning umumiy hadi $\frac{2n}{2n+1}$ ko‘rinishda bo‘ladi. U holda

$$\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+\ldots+\frac{2n}{2n+1}+\ldots,\ a_n=\frac{2n}{2n+1}.$$

Ma’lumki, qator yaqinlashuvchi bo‘lsa, uning umumiy hadi $n\to\infty$ da nolga intiladi. Buni e’tiborga olgan holda

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{2n}{2n+1}=\lim_{n\to\infty}\frac{2}{2+\frac{1}{n}}=1$$

Topshiriqlar.

Taqqoslash alomatlaridan foydalanib quyidagi qatorlarning yaqinlashishi yoki uzoqlashishini ko‘rsating.

1. $\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}$ Javob: yaqinlashuvchi

2. $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ Javob: uzoqlashuvchi

3. $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2\left(1+2^2\right)}}+\frac{1}{\sqrt{3\left(1+3^2\right)}}+\ldots+\frac{1}{\sqrt{n\left(1+n^2\right)}}+\ldots$ Javob: yaqinlashuvchi

4. $\sin\frac{\pi}{3}+\sin\frac{\pi}{9}+\ldots+\sin\frac{\pi}{3^n}+\ldots$ Javob: yaqinlashuvchi

5. $\frac{1}{\sqrt{1\cdot 2}}+\frac{1}{\sqrt{2\cdot 3}}+\frac{1}{\sqrt{3\cdot 4}}+\ldots+\frac{1}{\sqrt{n(n+1)}}+\ldots$ Javob: uzoqlashuvchi

6. $1+\frac{1+2}{1+2^2}+\frac{1+3}{1+3^2}+\ldots+\frac{1+n}{1+n^2}+\ldots$ Javob: uzoqlashuvchi

7. $\sin 1+\sin\frac{1}{2}+\ldots+\sin\frac{1}{n}+\ldots$ Javob: yaqinlashuvchi

8. $\ln\frac{3}{2}+\ln\frac{2^2+2}{2^2+1}+\ln\frac{3^2+2}{3^2+1}+\ldots+\ln\frac{n^2+2}{n^2+1}+\ldots$ Javob: yaqinlashuvchi

9. $2\sin\frac{\pi}{3}+2^2\sin\frac{\pi}{3^2}+\ldots+2^n\sin\frac{\pi}{3^n}+\ldots$ Javob: uzoqlashuvchi

10. $\frac{1}{1*2}+\frac{1}{2*3}+\frac{1}{n(n+1)}$; 11. $\frac{1}{1*3}+\frac{1}{3*5}+\ldots+\frac{1}{(2n-1)(2n+1)}+\ldots$

Quyidagi qatorlarinng yig’indilarini toping.

1. $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{3\cdot 2^{n-1}}+\ldots$ Javob: $\frac{2}{3}$

2. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots$ Javob: 1

3. $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\ldots++\frac{1}{n(n+1)(n+2)}+\ldots$ Javob: $\frac{1}{4}$

4. $\frac{3}{1^2\cdot 2^2}+\frac{5}{2^2\cdot 3^2}+\ldots+\frac{2n+1}{n^2(n+1)^2}+\ldots$ Javob: 1

5. $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\ldots$ Javob: $\frac{3}{4}$

6. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots$ Javob: $\frac{1}{2}$

Quyidagi qatorlarning yaqinlashishini zaruriy sharti yordamida tekshiring.

1. $\frac{2}{1}+\frac{5}{8}+\frac{10}{27}+\ldots+\frac{n^2+1}{n^2}+\ldots$ Javob: uzoqlashuvchi

2. $\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\ldots+\frac{2n-1}{2n}+\ldots$ Javob: uzoqlashuvchi

3. $\cos 1+\cos\frac{1}{2}+\cos\frac{1}{3}+\ldots+\cos\frac{1}{n}+\ldots$ Javob: uzoqlashuvchi

4. $\ln\frac{2}{1}+\ln\frac{3}{2}+\ln\frac{4}{3}+\ldots$ Javob: uzoqlashuvchi

5. $\frac{2}{3}+\frac{3}{5}+\frac{4}{7}+\ldots+\frac{n+1}{2n+1}+\ldots$ Javob: uzoqlashuvchi

7-MAVZU. MUSBAT HADLI SONLI QATORLAR YAQINLASHISHINING YETARLI SHARTLARI: DALAMBER ALOMATI, KOSHINING RADIKAL VA INTEGRAL ALOMATLARI. ISHORASI ALMASHINUVCHI VA O‘ZGARUVCHAN ISHORALI SONLI QATORLAR. LEYBNITS TEOREMASI. ABSOLYUT VA SHARTLI YAQINLASHISH

Faraz qilaylik, u_1, u_2, ... , u_n, ... biror musbat sonlar bo‘lsin. U vaqtda qator

$$u_1-u_2+u_3-u_4+...$$

ishorasi almashinuvchi qator deyiladi.

Teorema. Agar $u_1>u_2>u_3>...$bo‘lib, n ning har qanday natural qiymatida ham $u_n>0$ va $\lim\limits_{n\to\infty} u_n = 0$ bo‘lsa, u vaqtda $u_1-u_2+u_3-u_4+...$ (1) ishorasi navbatlashuvchi qator yaqinlashuvchi bo‘ladi, qatorning yig’indisi musbat son bo‘ladi va u qatorning birinchi hadidan katta bo‘lmaydi.

Ana shu teorema Leybnist teoremasi deyiladi.

Dalamber alomati

$a_1+a_2+\ldots+a_n+\ldots$, (n+1) hadining n – hadiga nisbati $n\to\infty$ da L chekli limitga ega bo‘lsa, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = L$ bo‘lsa, u vaqtda

1) $L \prec 1$ qator yaqin bo‘ladi.
2) $L \succ 1$ bo‘lgan holda qator uzoqlashadi.
3) L = 1 bo‘lgan holda D belgisi javob bera olmaydi.

Koshi alomati

Agar musbat hadli qator $a_1 + a_2 + \ldots + a_n + \ldots$, uchun $\sqrt[n]{a_n}$ miqdor $n\to\infty$ da L chekli limitga ega, ya’ni $\lim\limits_{n\to\infty}\sqrt[n]{u_n}$ = L bo‘lsa

1) L<1 qator yaqinlashuvchi bo‘ladi.
2) L>1 bo‘lgan holda qator uzoqlashadi.
3) L = 1 bo‘lgan holda qo‘shimcha tekshiriladi.

Integral alomati

1) $a_1 + a_2 + \ldots + a_n + \ldots$, musbat hadli va o‘suvchi bo‘lmasin, $a_1 \geq a_2 \geq \ldots$ va U(x) shunday o‘smaydigan uzluksiz bo‘lib, $U(1) = a_1$, $U(2) = a_2$ … $U(n) = a_n$ bo‘lsin,

a) $\int_1^{\infty} U(x)dx$ xosmas integral yaqinlashsa (1) qator ham yaqinlashadi.

b) integral uzoqlashsa (1) qator uzoqlashadi.

1. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

Yechish: Bu qatorning umumiy hadi $a_n = \frac{n}{3^n}$, keyingi hadi $a_{n+1} = \frac{n+1}{3^{n+1}}$. Dalamber alomatiga asosan,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \lim_{n\to\infty} \frac{(n+1)\cdot 3^n}{n\cdot 3^{n+1}} = \frac{1}{3}\lim_{n\to\infty} \frac{(n+1)}{n} = \frac{1}{3} < 1$$

bo‘lgani uchun berilgan qator yaqinlashuvchi bo‘ladi.

2. $\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n}$**.**

Yechish: Bu yerda $a_n = \frac{3^n \cdot n!}{n^n}$, $a_{n+1} = \frac{3^{n+1}\cdot (n+1)!}{(n+1)^{n+1}}$. Dalamber alomatiga asosan

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} : \frac{3^n \cdot n!}{n^n} = \lim_{n\to\infty} \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n \cdot n!} =$$

$= \lim_{n\to\infty} \frac{3n^n}{(n+1)^n} = 3\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 3\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{3}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{3}{e} > 1$ bo‘lgani uchun berilgan qator uzoqlashuvchi.

3-misol. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

Yechish: Koshining radikal alomatiga ko‘ra

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

bo‘lgani uchun qator yaqinlashuvchi.

4-misol. $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ (Dirixle qatori) qatorni yaqinlashishga tekshiring.

Yechish: Qatorni yaqinlashishga tekshirish uchun Koshining integral alomatidan foydalanamiz.

$f(x)=\frac{1}{x^{\alpha}}$ deb olib, ushbu $\int_{1}^{\infty}\frac{dx}{x^{\alpha}}$ xosmas integralni tekshiramiz.

$$\int_{1}^{\infty}\frac{dx}{x^{\alpha}}=\lim_{b\to\infty}\int_{1}^{b}x^{-\alpha}dx=\lim_{b\to\infty}\begin{cases}\frac{1}{1-\alpha}\left(b^{-\alpha}-1\right), & \alpha\neq 1\\ \ln b, & \alpha=1\end{cases}=$$

$$=\begin{cases}\frac{1}{\alpha-1}, & \text{agar } \alpha>1 \text{ bo'lsa,} \quad \text{yaqinlashuvchi}\\ \infty, & \text{agar } \alpha<1 \text{ bo'lsa,} \quad \text{uzoqlashuvchi}\\ \infty, & \text{agar } \alpha=1 \text{ bo'lsa,} \quad \text{uzoqlashuvchi}\end{cases}$$

Demak, xosmas integral $\alpha>1$ da yaqinlashuvchi, $\alpha\le 1$ da uzoqlashuvchi . Koshining integral alomatiga ko‘ra $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ qator $\alpha>1$ da yaqinlashuvchi, $\alpha\le 1$ da uzoqlashuvchi bo‘ladi. $\alpha=1$ bo‘lganda $\sum_{n=1}^{\infty}\frac{1}{n}$ garmonik qator hosil bo‘ladi.

Agar qatorning hadlari orasida musbatlari ham, manfiylari ham bo‘lsa, bunday qator o‘zgaruvchan ishorali qator deyiladi. Ishorasi navbatlashuvchi qatorlar esa ishorasi o‘zgaruvchi qatorlarning xususiy holidir. Faraz qilaylik $u_1,u_2,...,u_n,...$ sonlarning biror cheksiz ketma-ketligi berilgan bo‘lsin.

1-Teorema. Agar $u_1-u_2+u_3-u_4+...u_n+$ ... (1) o‘zgaruvchan ishorali qator hadlarining absolyut qiymatlaridan tuzilgan

$$|u_1|+|u_2|+|u_3|+...+|u_n|+... \qquad (2)$$

qator yaqinlashuvchi bo‘lsa, u vaqtda (1) o‘zgaruvchan ishorali qator ham yaqinlashuvchi bo‘ladi.

Ta’rif. Ushbu o‘zgaruvchan ishorali qator $u_1+u_2+u_3+...+u_4+...$ hadlarining absolyut qiymatlaridan tuzilgan

$$|u|+|u_2|+|u_3|+...+|u_n|+...$$

qator yaqinlashsa, berilgan qator absolyut yaqinlashuvchi deyiladi.

Agar (1) o‘zgaruvchan ishorali (1) qator shartli yoki noabsolyut

yaqinlashuvchi qator deb ataladi.

1-Misol. Ushbu $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$qator shartli yaqinlashuvchi qatordir. (Leybnits teoremasiga asosan). Ammo bu qator absolyut yaqinlashuvchi emas, chunki qator hadlarining absolyut qiymatlaridan tuzilgan $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ qator garmonik qator bo‘lib, u ma’lumki uzoqlashuvchidir.

Ammo $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\ldots$ qator absolyut yaqinlashuvchi qatordir, chunki $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$ qator yaqinlashuvchidir.

2-teorema. Agar qator absolyut yaqinlashuvchi bo‘lsa, uning hadlarining o‘rinlarini ixtiyoriy ravishda almashtirganda ham, u absolyut yaqinlashuvchanligicha qoladi. Bu holda qatorning yig’indisi qator hadlarining tartibiga bog’liq bo‘lmaydi.

Bu xossa shartli yaqinlashuvchi qatorlar uchun o‘z kuchini yo‘qotadi.

3-teorema. Agar qator shartli yaqinlashsa, ixtiyoriy ravishda olingan A soni qanday bo‘lishidan qat’iy nazar, bu qatorning hadlarini qatorning yig’indisi shu A sonining o‘ziga teng bo‘ladigan qilib almashtirish mumkin. Shu bilan birga shatrli yaqinlashuvchi qator hadlarining o‘rinlarini shunday almashtirish mumkinki, bu o‘rin almashtirishdan keyin hosil bo‘lgan qator uzoqlashuvchi bo‘lib qoladi.

2-Misol. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ o‘zgaruvchan ishorali qator noabsolyut yaqinlashadi. Uning yig’indisi S bo‘lsin. Ma’lumki, S>0. Bu qatorni $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\ldots+\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}+\ldots(**)$ shaklda yozamiz.

Bu oxirgi qator yig’indisini topamiz. Uni S_1 desak $S_1=(\frac{1}{2}-\frac{1}{4})+(\frac{1}{6}-\frac{1}{8})+\ldots+(\frac{1}{4k-2}-\frac{1}{4k})+\ldots+=\frac{1}{2}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)+\ldots\right]=$

$\frac{1}{2}\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)+\ldots\right]=\frac{1}{2}\cdot S$ hosil bo‘ladi. Demak, bu qaralgan holda qatorning yig’indisi, uning hadlari o‘rinlarini almashtirgandan keyin o‘zgardi (ikki marta kamaydi).

Agar ishorali navbat- chi $u_1-u_2+u_3-u_4+\ldots(u_n \geq 0)$ (1)- qatorning hadlari. $u_1 > u_2 > u_3 > \ldots$ va $\lim_{n\to\infty} u_n=0$ bo‘lsa, (1) qator yaqinlashadi. Uning yig’indisi musbat bo‘ladi. Va 1 – haddan katta bo‘lmaydi. Qatorning hadlaridan tuzilgan $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ absalyut qator yaqinlashsa (1) – qator albatta yaqinlashadi va u absalyut yaqinlashadi deyiladi. Qator absolyut yaqinlashmasligi mumkin. lekin ishorasi almashib keluvchi qator yaqinlashuvchi bo‘lishi mumkin.

Topshiriqlar.

Quyida qatorlarning yaqinlashishga tekshiring:

1. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+(-1)^{n+1}\frac{1}{2n-1}+\ldots$
2. $-\frac{1}{3^2}+\ldots+(-1)^{n+1}\frac{1}{(2n-1)^2}+\ldots$
3. $\frac{1}{\ln^2}-\frac{1}{\ln^3}+\ldots+(-1)^{n+1}\frac{1}{\ln(n+1)}+\ldots$

Dalamber va Koshi alomatlaridan foydalanib quyidagi qatorlarning yaqinlashish yoki uzoqlashishini tekshiring.

1. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}+\ldots$ Javob: yaqinlashuvchi
2. $\frac{5}{1!}+\frac{5^2}{2!}+\frac{5^3}{3!}+\ldots+\frac{5^n}{n!}+\ldots$ Javob: uzoqlashuvchi
3. $1+\frac{2^2}{2!}+\frac{3^3}{3!}+\ldots+\frac{n^n}{n!}+\ldots$ Javob: uzoqlashuvchi
4. $2+\left(\frac{2+1}{2\cdot 2-1}\right)^2+\left(\frac{3+1}{2\cdot 3-1}\right)^3+\ldots+\left(\frac{n+1}{2n-1}\right)^n+\ldots$ Javob: yaqinlashuvchi
5. $\frac{2}{3}+\frac{\left(\frac{3}{2}\right)^4}{9}+\ldots+\frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}+\ldots$ Javob: yaqinlashuvchi
6. $2+\frac{2\cdot 5}{1\cdot 5}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\ldots++\frac{2\cdot 5\cdot 8\cdot\ldots\cdot(3n-1)}{1\cdot 5\cdot 9\cdot\ldots\cdot(4n-3)}+\ldots$ Javob: yaqinlashuvchi
7. $1+\frac{1\cdot 4}{3!!}+\frac{1\cdot 4\cdot 7}{5!!}+\ldots++\frac{1\cdot 4\cdot 7\cdot\ldots\cdot(3n-2)}{(2n-1)!!}+\ldots$ Javob: uzoqlashuvchi

8. $\sin\frac{\pi}{2}+\sin^2\frac{\pi}{4}+\ldots+\sin^n\frac{\pi}{2n}+\ldots$ Javob: yaqinlashuvchi

9. $\frac{3}{2^{-1}}+\frac{3^3}{2^2}+\frac{3^5}{2^5}+\ldots+\frac{3^{2n+1}}{2^{3n-1}}+\ldots$ Javob: uzoqlashuvchi

10. $\frac{1}{\sqrt{3}}+\frac{5}{\sqrt{2\cdot 3^2}}+\frac{9}{\sqrt{3\cdot 3^3}}+\ldots+\frac{4n-3}{\sqrt{n\cdot 3^n}}+\ldots$ Javob: yaqinlashuvchi

Koshining integral alomatidan foydalanib, quyidagi qatorlarning yaqinlashishini tekshiring.

1. $\frac{1}{2(\ln 2)^2}+\frac{1}{3(\ln 3)^2}+\ldots+\frac{1}{n(\ln n)^2}+\ldots$ Javob: yaqinlashuvchi

2. $1+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{9}}+\ldots+\frac{1}{\sqrt{4n+1}}+\ldots$ Javob: uzoqlashuvchi

3. $\frac{2}{3+1^2}+\frac{2}{3+2^2}+\frac{2}{3+3^2}+\ldots+\frac{2}{3+n^2}+\ldots$ Javob: yaqinlashuvchi

4. $\left(\frac{1+1}{1+1^2}\right)^2+\left(\frac{1+2}{1+2^2}\right)^2+\ldots+\left(\frac{1+n}{1+n^2}\right)^2+\ldots$ Javob: yaqinlashuvchi

5. $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ Javob: yaqinlashuvchi

6. $\frac{e^{-\sqrt{1}}}{\sqrt{1}}+\frac{e^{-\sqrt{2}}}{\sqrt{2}}+\frac{e^{-\sqrt{3}}}{\sqrt{3}}+\ldots+\frac{e^{-\sqrt{n}}}{\sqrt{n}}+\ldots$ Javob: yaqinlashuvchi

Quyidagi qatorlarni yaqinlashishga tekshiring.

1. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$ Javob: uzoqlashuvchi

2. $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$ Javob: yaqinlashuvchi

3. $\frac{1}{2^3}+\frac{2}{3^3}+\frac{3}{4^3}+\ldots+\frac{n}{(n+1)^3}+\ldots$ Javob: yaqinlashuvchi

4. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n(n-1)}}$ Javob: uzoqlashuvchi

5. $\sum_{n=1}^{\infty}\frac{n}{2n-1}$ Javob: uzoqlashuvchi

6. $\sum_{n=1}^{\infty}\frac{n!}{3^{n-1}}$ Javob: uzoqlashuvchi

7. $\sum_{n=1}^{\infty}\frac{2n-1}{(\sqrt{3})^n}$ Javob: yaqinlashuvchi

8. $\sum_{n=1}^{\infty}\frac{n^2}{2n^2+1}$ Javob: uzoqlashuvchi

9. $\sum_{n=1}^{\infty}\frac{n^3}{e^n}$ Javob: yaqinlashuvchi

10. $\sum_{n=1}^{\infty}\left(\frac{1+n}{1+n^2}\right)^2$ Javob: yaqinlashuvchi

11. $\sum_{n=0}^{\infty}\frac{1}{\sqrt{4n+1}}$ Javob: uzoqlashuvchi

12. $\sum_{n=3}^{\infty}\frac{n}{n^4-9}$ Javob: yaqinlashuvchi

13. $\sum_{n=1}^{\infty}\frac{2}{2^n}\left(\frac{n+1}{n}\right)^{n^2}$ Javob: uzoqlashuvchi

Ishorali takrorlanuvchi qatorning absalyut va shartli yaqinlashishini tekshiring:

1. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+1}\left(2+\frac{1}{n}\right)$

2. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$

3. $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$

4. $\sum_{n=1}^{\infty}\frac{5n^2+n-1}{(4n^2-1)(n2+n)}$

5. $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$

6. $\sum_{n=1}^{\infty}arctg\,\frac{1}{2n^2}$

7. $\sum_{n=1}^{\infty}(\sqrt[3]{n+2}-2\sqrt[3]{n+1}+\sqrt[3]{n})$

8. $\sum_{n=1}^{\infty}\ln^{2n}2$

9. $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$

10. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)(n+3)}$

8-MAVZU. FUNKSIONAL QATORLARNING YAQINLASHISH SOHASI. DARAJALI QATORLAR. YAQINLASHISH RADIUSI. QATORLARNI DIFFERENSIALLASH VA INTEGRALLASH.

Agar $u_1+u_2+u_3+...+u_n+...$ qatorning hadlari x ning funksiyalari bo‘lsa, bu qator funksional qator deyiladi. Ushbu

$$u_1(x)+u_2(x)+...+u_n(x)+.. \quad (1)$$

funksional qatorni qaraymiz. Bunda x ning turli qiymatlarida turli yaqinlashuvchi va uzoqlashuvchi qatorlar hosil bo‘lishi mumkin. x ning funksional qator yaqinlashadigan qiymatlari to‘plami shu qatorning yaqinlashish sohasi deyiladi. Qatorning yaqinlashish sohasidagi yig’indisi x ning biror funksiyasidir. Shu sabab funksional qator yig’indisi $S(x)$ orqali belgilanadi.

Misol. $1+x+x^2+...+x^{n-1}+...$ funksional qator x ning $|x|<1$ tengsizlikni qanoatlantiruvchi qiymatlarida yaqinlashadi va x ning bu qiymatlarida qator yig‘indisi $\frac{1}{1-x}$ ga teng bo‘ladi. Demak, (-1;1) oraliqda $\frac{1}{1-x}=1+x+x^2+...+x^{n-1}$ bo‘ladi. Shunday qilib, bu qator yig’indi $S(x)\frac{1}{1-x}$ funksiyani aniqlaydi.

Agar (1) qatorning dastlabki n ta hadi yig’indisini $S_n(x)$ bilan, qator yig’indisini $S(x)$ bilan va ushbu $U_{n+1}(x)+U_{n+2}(x)+\dots$ ni qator yig’indisi $r_n(x)$ bilan belgilasak, $S(x)=S_n(x)+r_n(x)$ bo‘ladi.

Demak, $r_n(x)=S(x)-S_n(x)$ bo‘ladi va $r_n(x)$ (1) qatorning Qoldig’i deyiladi. Qatorning yaqinlashish sohasidagi barcha x lar uchun $\lim_{n\to\infty}(x)=S(x)$ bo‘lgani uchun x ning bunday qiymatlarida $\lim_{n\to\infty} r_n(x)=\lim_{n\to\infty}\left[S(x)-S_n(x)\right]=0$ bo‘ladi, ya’ni yaqinlashuvchi qatorning $r_n(x)$ qoldig’i oldingi $n\to\infty$ da nolga intiladi.

Ta’rif. Agar hadlari musbat bo‘lgan shunday

$$\alpha_1+\alpha_2+...+\alpha_n+... \quad (2)$$

sonli yaqinlashuvchi qator mavjud bo‘lib, x ning berilgan sohadan barcha qiymatlari uchun:

$$|u_1(x)\le\alpha_1|,|u_2(x)\le\alpha_2|,...,|u_n(x)\le\alpha_n|,... \quad (3)$$

munosabat bajarilsa

$$u_1(x)+u_2(x)+...+u_n(x)+... \qquad (4)$$

funksional qator x ning biror o‘zgarish sohasida kuchaytirilgan qator deb ataladi.

1-ta’rif. Ushbu $a_0+a_1x+a_2x^2+...+a_nx^n+...$ (1) funksional qator darajali qator deyiladi, bunda $a_0, a_1, a_2, ... a_n, ...$ o‘zgarmas sonlar bo‘lib, ular qator koeffitsientlari deyiladi.

Darajali qatorning yaqinlashish sohasi biror oraliq (interval) dan iborat bu oraliq ba’zan nuqtaga aylanishi mumkin. Juda muhim quyidagi teoremani qaraymiz.

1-teorema (Abel teoremasi)

1) Agar darajali qator noldan farqli biror x_0 ($x_0\neq0$) qiymatda yaqinlashsa, x ning $|x|<|x_0|$ tengsizlikni qanoatlantiruvchi har qanday qiymatlarida u absolyut yaqinlashadi;

2) agar qator biror $x`_0$ qiymatda uzoqlashsa, x ning $|x|>|x`_0|$ tengsizlikni qanoatlantiruvchi har bir qiymatida qator uzoqlashadi.

Quyidagi qatorlarning yaqinlashish radiuslari va yaqinlashish intervallarini toping. Yaqinlashish intervalining chegaralarida qatorning yaqinlashishini tekshiring.

1-misol. $\sum_{n=1}^{\infty}\frac{2^n x^n}{\sqrt{3^n}}$.

Yechish: Bu yerda $a_n=\frac{2^n x^n}{\sqrt{3^n}}$. Qatorning yaqinlashish radiusini quyidagi Koshi alomati yordamida topamiz.

$$R=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{a_n}}=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{\frac{2^n}{\sqrt{3^n}}}}=\frac{1}{\frac{2}{\sqrt{3}}}=\frac{\sqrt{3}}{2},\ R=\frac{\sqrt{3}}{2}.$$

Demak, yaqinlashish intervali $\left(-\frac{\sqrt{3}}{2};\frac{\sqrt{3}}{2}\right)$ bo‘ladi. Intervalning chegaralarida qatorning yaqinlashishini alohida tekshiramiz.

$x=-\frac{\sqrt{3}}{2}$ bo‘lsa $\sum_{n=1}^{\infty}(-1)^n$ uzoqlashuvchi qator,

$x=\frac{\sqrt{3}}{2}$ bo‘lsa $\sum_{n=1}^{\infty}1^n=1+1+1+\ldots+1+\ldots$ uzoqlashuvchi qator hosil bo‘ladi. Demak berilgan qatorning yaqinlashish intervali $\left(-\frac{\sqrt{3}}{2};\frac{\sqrt{3}}{2}\right)$ dan iborat.

2-misol. $\sum_{n=1}^{\infty}\frac{x^n}{n\cdot 2^n}$.

Yechish: n - hadining koeffitsienti $a_n=\frac{1}{n\cdot 2^n}$. Qatorning yaqinlashish radiusini topish formulasiga asosan topamiz:

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{(n+1)\cdot 2^{n+1}}{n\cdot 2^n}=\lim_{n\to\infty}\frac{2n+2}{n}=2,$$

$R=2$ va yaqinlashish intervali $(-2;2)$ bo‘ladi. Interval chegaralarida, ya’ni $x=-2$ va $x=2$ nuqtalarda tekshiramiz.

$x=-2$ bo‘lganda $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ qator hosil bo‘ladi va u Leybnist teoremasi shartlarini qanoatlantiradi. Demak, qator $x=-2$ nuqtada yaqinlashadi.

$x=2$ bo‘lganda $\sum_{n=1}^{\infty}\frac{1}{n}$ uzoqlashuvchi garmonik qator hosil bo‘ladi. Shunday qilib, berilgan darajali qator $[-2;2)$ yarim oraliqda yaqinlashuvchi.

3-misol. $\sum_{n=1}^{\infty}\frac{4^n}{n^3\left(x^2-4x+7\right)^n}$.

Yechish: Qatorning n-hadi va $(n+1)$- hadini yozamiz:

$$u_n(x)=\frac{4^n}{n^3\left(x^2-4x+7\right)^n};\ u_{n+1}(x)=\frac{4^{n+1}}{(n+1)^3\left(x^2-4x+7\right)^{n+1}}.$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{4^{n+1}}{(n+1)^3\left(x^2-4x+7\right)^{n+1}}\cdot\frac{n^3\left(x^2-4x+7\right)^n}{4^n}\right|=$$

$$=\lim_{n\to\infty}\left|\frac{4n^3}{(n+1)^3\left(x^2-4x+7\right)}\right|=\frac{4}{x^2-4x+7}<1$$

Bu tengsizlikni yechib yaqinlashish oralig’ini topamiz.

$$x\in(-\infty,1)\cup(3,\infty)$$

Endi qatorni chegaradagi nuqtalarda yaqinlashishini tekshiramiz. $x=1$ da

$$\sum_{n=1}^{\infty}\frac{1}{n^3}$$

qator hosil bo‘ladi. Bu garmonik qatorning umumlashgan $p=3$ bo‘lgani uchun yaqinlashuvchidir $x=3$ da

$$\sum_{n=1}^{\infty}\frac{1}{n^3}$$

hosil bo‘ladi. Bu qator ham yaqinlashuvchi. Demak, yaqinlashish oralig‘i

$$(-\infty, 1]\cup[3, \infty).$$

Topshiriqlar.

Quyidagi qatorlarning yaqinlashish radiuslari va yaqinlashish intervallarini toping. Yaqinlashish intervalining chegaralarida qatorning yaqinlashishini tekshiring.

1. $\sum_{n=1}^{\infty}x^n$ Javob: $R=1,\ (-1;1)$

2. $\sum_{n=1}^{\infty}\frac{(x-4)^n}{n}$ Javob: $R=1,\ [3;5]$

3. $\sum_{n=1}^{\infty}n!\cdot x^n$ Javob: $R=0,\ x=0$

4. $\sum_{n=1}^{\infty}(-2)^n x^{2n}$ Javob: $R=\frac{1}{\sqrt{2}},\ \left(-\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right)$

5. $\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$ Javob: $R=1,\ [-1;1]$

6. $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ Javob: $R=1,\ [-1;1]$

7. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^{2n}}{n\cdot 4^n}$ Javob: $R=2,\ [0;4]$

8. $\sum_{n=1}^{\infty}\frac{n}{n+1}\left(\frac{x}{2}\right)^n$ Javob: $R=2,\ (-2;2)$

9. $\sum_{n=1}^{\infty}\frac{n^5}{(n+1)!}(x+5)^{2n+1}$ Javob: $R=\infty,(-\infty;\infty)$

10. $\sum_{n=1}^{\infty}\frac{10^n x^n}{\sqrt{n}}$ Javob: $R=\frac{1}{10},\ (-0{,}1;0{,}1)$

11. $\sum_{n=1}^{\infty}\frac{(2x+1)^n}{3n-2}$ Javob: $R=1,[-1;0)$

12. $\sum_{n=1}^{\infty} \frac{(-x)^n}{3^{n-1}\sqrt{n}}$ Javob: $R=3,\ (-3;\ 3]$

13. $\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{(2n)!} x^{2n}$ Javob: $R=\infty,\ (-\infty;\ \infty)$

14. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} x^n$ Javob: $R=\frac{1}{e},\ \left(-\frac{1}{e};\ \frac{1}{e}\right)$

9-MAVZU. FUNKSIYALARNI TEYLOR VA MAKLOREN QATORLARIGA YOYISH. BINOMIAL QATOR. ASOSIY ELEMENTAR FUNKSIYALARNI QATORLARGA YOYISH. FUR'E QATORI VA FUR'E KOEFFITSIYENTLARI. FUR'E QATORINING YAQINLASHISHI.

Teylor va Makloren qatorlari. x=a nuqta atrofida (n+1)-tartibli hamma hosilalarga ega bo'lgan f (x) uchun Teylorning quyidagi formulasini bilamiz:

$$f(x)=f(a)+\frac{x-a}{1}f'(a)+\frac{(x-a)^2}{1\cdot 2}f''(a)+...+\frac{(x-a)^n}{n!}f^{(n)}(a)+R_n(x) \qquad (1)$$

Bu yerda, $R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)]$, $0<\theta<1$

Teylor formulasi qoldiq hadining Lagranj formulasi.

Agar f (x) funksiya x=a nuqta atrofida barcha hosilalarga ega bo'lsa, $n\to\infty$ da qoldiq had R_n uchun $\lim\limits_{n\to\infty}R_n(x)=0$ bo'ladi.

$$f(x)=f(0)+\frac{x}{1}f'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^n}{n!}f^{(n)}(0)+...$$

Makloren qatorlari. x=a nuqta atrofida (n+1)-tartibli hamma hosilalarga ega bo'lgan f (x) uchun Teylorning quyidagi formulasini bilamiz:

$$f(x)=f(a)+\frac{x-a}{1}f'(a)+\frac{(x-a)^2}{1\cdot 2}f''(a)+...+\frac{(x-a)^n}{n!}f^{(n)}(a)+R_n(x) \qquad (1)$$

Bu yerda $R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)]$, $0<\theta<1$

Teylor formulasi qoldiq hadining Lagranj formulasi.

Agar f (x) funksiya x=a nuqta atrofida barcha hosilalarga ega bo'lsa, $n\to\infty$ da qoldiq had R_n uchun $\lim\limits_{n\to\infty}R_n(x)=0$ bo'ladi.

$$f(x)=f(0)+\frac{x}{1}f'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^n}{n!}f^{(n)}(0)+... \qquad (2)$$

Ba'zi funksiyalarni Makloren qatoriga yoyish.

1) f (x)=sinx bo'lsin. Bu funksiyani Makloren qatoriga yoyamiz. Ma'lumki

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} + R_{2n}(x)$$

$\lim_{n\to\infty} R_{2n}(x) = 0$ bo‘lgani uchun bu formuladan quyidagi qator hosil bo‘ladi

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} + \qquad (1)$$

Bu qatordan x turli qiymatlar olganda sinx ning qiymatlarini hisoblash uchun foydalaniladi.

Masalan, sin 10^0 ni 10^{-5} gacha aniqlik bilan hisoblaylik. 10^0 yoki radian hisobida, $\frac{\pi}{18} \approx 0{,}174533$ bo‘lgani uchun,

$$\sin 10^0 = \frac{\pi}{18} - \frac{1}{31}\left(\frac{\pi}{18}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{18}\right)^5 - \frac{1}{7!}\left(\frac{\pi}{18}\right)^7 + \ldots$$

Agar birinchi ikkita had bilan chegaralansak, $\frac{\pi}{18} \approx 0{,}173647$ hosil bo‘ladi. Bu yerda birinchi to‘rtta raqam to‘g’ridir.

2) Xuddi shuning kabi f (x)=e^x quyidagini hosil qilish mumkin.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots, \qquad (2)$$

hamda

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots \qquad (3)$$

Xuddi shuning kabi f (x)=cosx funksiya uchun,

f (x)=$(1+x)^m$ funksiyani qaraymiz. Bu yerda m-ixtiyoriy o‘zgarmas son.

Bu funksiya (1+x) f '(x)=m f (x) (4) differensial tenglamani va f (0)=1 boshlang’ich shartni qanoatlantiradi.

F(x)=1+a_1x+a_2x^2+. . .a_nx^n+. . . (5) darajali qatorni yozish mumkin. Buni (4) tenglamaga qo‘ysak,

(1+x))(a_1+2a_2x+3a_3x^2+ . . .+na_nx^{n-1}+. . .)=m(1+a_1x+a_2x^2+. . .+a_nx^n+. . .) hosil bo‘ladi.

Tenglikning turli qismlaridagi bir xil darajali x larning koeffitsientlarini tenglab, quyidagilarni topamiz:

$a_1=m,\ a_1+2a_2=ma_1,\ldots,na_n+(n+1)a_{n+1}=ma_n,\ldots$

bulardan,

$$a_0=1,\ a_1=m,\quad a_2=\frac{a_1(m-1)}{2}=\frac{m(m-1)}{2},\qquad a_3=\frac{a_2(m-2)}{3}=\frac{m(m-1)(m-2)}{2\cdot 3}\ldots$$

$$a_n=\frac{m(m-1)\ldots[m-(n-1)]}{1\cdot 2\cdot\ldots\cdot n}\qquad (6)$$

Bular binomial keffisientlardir. Ularni (5) formulaga qo‘ysak:

$$f(x)=1+mx+\frac{m(m-1)}{1\cdot 2}x^2+\ldots+\frac{m(m-1)\ldots[m-(n-1)]}{1\cdot 2\cdot\ldots n}x^n+\ldots \qquad (7)$$

bu yerda

$$u_{n+1}=\frac{m(m-1)\ldots[m-n+1]}{n!}x^m,\qquad u_n=\frac{m(m-1)\ldots[m-n+2]}{(n-1)!}x^{n-1},$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{m(m-1)\ldots(m-n+1)(n-1)!}{m(m-n)\ldots(m-n+2)n!}x\right|=\lim_{n\to\infty}\left|\frac{m-n+1}{n}\right||x|=|x|$$

Shunday qilib, (7) qator |x|<1 bo‘lganda yaqinlashadi.

Demak,

$$(1+x)^m=1+mx+\frac{m(m-1)}{1\cdot 2}x^2+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3+\ldots \qquad (8)$$

jumladan m=-1 bo‘lganda:

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots \qquad (9)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots \qquad (10)$$

hosil qilish mumkin.

Binomial qatorlar

m=-1/2 bo‘lganda:

$$\frac{1}{\sqrt{1+x}}=1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4-\ldots \qquad (1)$$

Binom yoyilmasini boshqa funksiyalarning yoyilmasiga tadbiq etamiz:

f (x)=arcsinx funksiyani Makloren qatoriga yoyamiz. (1) tenglikdagi x o‘rniga $-x^2$ ifodani qo‘ysak:

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^5+\ldots+\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}x^{2n}+\ldots$$

|x|<1 bo‘lganda, darajali qatorlarni integrallash haqidagi teoremaga asosan quyidagini hosil qilamiz:

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = \arcsin x = x + \frac{1\cdot x^3}{2\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \ldots \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\cdot\frac{x^{2n+1}}{2n+1} + \ldots$$

Bu qator (-1; 1) oraliqda yaqinlashadi. Qator x=±1 bo‘lganda ham yaqinlashishini va bu qiymatlar uchun qatorning yig’indisi arcsinx ga tengligini isbot qilish mumkin. U vaqtda x=1 deb olib, arcsin1 ni hisoblashning quyidagi formulasini hosil qilamiz:

$$\arcsin 1 = \frac{\pi}{2} = 1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7} + \ldots$$

1-misol. $f(x)=\sin\frac{\pi x}{4}$ funksiyani $x=2$ nuqta atrofida Teylor qatoriga yoying.

Yechish. Berilgan funksiyaning hosilalarini va $x=2$ nuqtadagi qiymatlarini topamiz.

$$f(x)=\sin\frac{\pi x}{4},\quad f(2)=\sin\frac{2\pi}{4}=1,$$

$$f'(x)=\frac{\pi}{4}\cos\frac{\pi x}{4}=\frac{\pi}{4}\sin\left(\frac{\pi x}{4}+\frac{\pi}{2}\right),\ f'(2)=\sin\left(\frac{2\pi}{4}+\frac{\pi}{2}\right)=0,$$

$$f''(x)=-\frac{\pi^2}{4^2}\sin\frac{\pi x}{4}=\frac{\pi^2}{4^2}\sin\left(\frac{\pi x}{4}+2\cdot\frac{\pi}{2}\right),$$

$$f''(2)=\frac{\pi^2}{4^2}\sin\left(\frac{2\pi}{4}+\pi\right)=-\frac{\pi^2}{4^2},$$

$$f'''(x)=-\frac{\pi^3}{4^3}\cos\frac{\pi x}{4}=\frac{\pi^3}{4^3}\sin\left(\frac{\pi x}{4}+3\cdot\frac{\pi}{2}\right),$$

$$f'''(2)=\frac{\pi^3}{4^3}\sin\left(\frac{2\pi}{4}+\frac{3\pi}{2}\right)=0,$$

$$f^{IV}(x)=\frac{\pi^4}{4^4}\sin\frac{\pi x}{4}=\frac{\pi^4}{4^4}\sin\left(\frac{\pi x}{4}+4\cdot\frac{\pi}{2}\right),$$

$$f^{IV}(2)=\frac{\pi^4}{4^4}\sin\left(\frac{2\pi}{4}+2\pi\right)=\frac{\pi^4}{4^4},$$

$$f^{(2k)}(x)=\frac{\pi^{2k}}{4^{2k}}\sin\left(\frac{\pi x}{4}+2k\cdot\frac{\pi}{2}\right),$$

$$f^{(2k)}(2)=\frac{\pi^{2k}}{4^{2k}}\sin\left(\frac{2\pi}{4}+k\pi\right)=(-1)^k\frac{\pi^{(2k)}}{4^{(2k)}},\qquad f^{(2k+1)}(x)=\frac{\pi^{2k+1}}{4^{2k+1}}\sin\left(\frac{\pi x}{4}+(2k+1)\cdot\frac{\pi}{2}\right),$$

$$f^{(2k+1)}(2)=\frac{\pi^{2k+1}}{4^{2k+1}}\sin\left(\frac{2\pi}{4}+\frac{(2k+1)\pi}{2}\right)=0.$$

Topilganlarni Teylor qatoriga qo‘ysak, quyidagiga ega bo‘lamiz

$$\sin\frac{\pi x}{4}=1-\frac{\pi^2}{4^2}\frac{(x-2)^2}{2!}+\frac{\pi^4}{4^4}\frac{(x-2)^4}{4!}-\ldots++\ldots+(-1)^k\frac{\pi^{2k}}{4^{2k}}\frac{(x-2)^{2k}}{(2k)!}+\ldots.$$

Bu darajali qatorning yaqinlashish intervalini Dalamber alomati yordamida topamiz.

$$\lim_{k\to\infty}\left|\frac{u_{2k+2}(x)}{u_{2k}(x)}\right|=\lim_{k\to\infty}\left|\frac{\pi^{2k+2}}{4^{2k+2}}\frac{(x-2)^{2k+2}}{(2k+2)!}\cdot\frac{4^{2k}}{\pi^{2k}}\frac{(2k)!}{(x-2)^{2k}}\right|=$$

$$=\frac{\pi^2}{4^2}(x-2)^2\lim_{k\to\infty}\frac{1}{(2k+1)(2k+2)}=0<1$$

bo‘lgani uchun x ning barcha qiymatlarida yuqorida berilgan qator yaqinlashuvchi, ya’ni qatorning yaqinlashish intervali $(-\infty;\infty)$ dan iborat. Endi Teylor formulasidan

$$R_{k+1}=\frac{\pi^{2k+1}}{4^{2k+1}}\frac{(x-2)^{2k+1}}{(2k+1)!}\sin\left(\frac{\pi\xi}{4}+(2k+1)\frac{\pi}{2}\right)$$

qoldiq hadni tekshiramiz. Har qanday k va ξ uchun $\left|\sin\left(\frac{\pi\xi}{4}+(2k+1)\frac{\pi}{2}\right)\right|\le 1$

$\lim_{k\to\infty}\left(\frac{\pi}{4}\right)^{2k+1}=0$ o‘rinli. Har qanday chekli x uchun

$$\lim_{k\to\infty}\frac{(x-2)^{2k+1}}{(2k+1)!}=0.$$

U holda $\lim_{k\to\infty}R_{2k+1}(x)=0$. Demak, berilgan $f(x)=\sin\frac{\pi x}{4}$ funksiya ixtiyoriy tartibdagi hosilaga ega va $\lim_{k\to\infty}R_{2k+1}(x)=0$ bo‘lgani uchun berilgan funksiya $(x-2)$ ning darajalari bo‘yicha Teylor qatoriga yoyiladi.

$$\sin\frac{\pi x}{4}=1-\frac{\pi^2}{4^2}\frac{(x-2)^2}{2!}+\frac{\pi^4}{4^4}\frac{(x-2)^4}{4!}-\ldots+(-1)^k\frac{\pi^{2k}}{4^{2k}}\frac{(x-2)^{2k}}{(2k)!}+\ldots.$$

2-misol. $f(x)=\frac{1}{x}$ funksiyani $x=-3$ nuqta atrofida Teylor qatoriga yoying.

Yechish. Berilgan $f(x)=\frac{1}{x}$ funksiya hosilalarini va ularning $x=-3$ nuqtadagi qiymatlarini topamiz.

$$f(x)=\frac{1}{x}=x^{-1},\qquad f(-3)=-\frac{1}{3},$$

$$f'(x)=-1\cdot x^{-2}=-\frac{1}{x^2},\qquad f'(-3)=-\frac{1}{3^2},$$

$$f''(x)=1\cdot 2\cdot x^{-3}=\frac{2!}{x^3},\qquad f''(-3)=-\frac{2!}{3^3},$$

$$f'''(x)=-1\cdot 2\cdot 3\cdot x^{-4}=-\frac{3!}{x^4},\qquad f'''(-3)=-\frac{3!}{3^4},$$

$$f^{(n)}(x)=(-1)^n\frac{n!}{x^{n+1}},\qquad f^{(n)}(-3)=-\frac{n!}{3^{n+1}},$$

Topilganlarni Teylor qatoriga qo‘ysak, $f(x)=\frac{1}{x}$ funksiya uchun Teylor qatori

$$-\frac{1}{3}-\frac{1!}{3^2}\frac{x+3}{1!}-\frac{2!}{3^4}\frac{(x+3)^2}{2!}-\ldots-\frac{n!}{3^{n+1}}\frac{(x+3)^n}{n!}-\ldots=$$
$$=-\frac{1}{3}\left(1+\frac{x+3}{3}+\frac{(x+3)^2}{3^2}+\ldots+\frac{(x+3)^n}{3^n}+\ldots\right).$$

ko‘rinishda bo‘ladi. Bu qatorning yaqinlashish intervalini Dalamber alomati bo‘yicha topamiz.

$$u_n(x)=\frac{(x+3)^n}{3^n};\ u_{n+1}(x)=\frac{(x+3)^{n+1}}{3^{n+1}}.$$

$$l=\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right|=\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{(x+3)^n}\right|==\lim_{n\to\infty}\frac{|x+3|}{3}=\frac{|x+3|}{3}<1.$$

Bundan $|x+3|<3$, $-3<x+3<3$, $-6<x<0$. Intervalning chegaralarida, ya’ni $x=-6$, $x=0$ da mos ravishda

$$1-1+1-1+\ldots,\qquad 1+1+1+1+\ldots$$

uzoqlashuvchi qatorlar hosil bo‘ladi. Demak, qatorning yaqinlashish intervali $(-6;\ 0)$ bo‘ladi. Hosil bo‘lgan qatorda

$$\lim_{n\to\infty}R_n(x)=0$$

ekanligini oson isbotlash mumkin. Demak, quyidagi tenglik o‘rinli:

$$\frac{1}{x}=-\frac{1}{3}\left(1+\frac{x+3}{3}+\frac{(x+3)^2}{3^2}+\ldots+\frac{(x+3)^n}{3^n}+\ldots\right).$$

3-misol. $f(x)=\frac{1}{x^2-3x+2}$ funksiyani x ning darajalari bo‘yicha Makloren qatoriga yoying.

Yechish. Berilgan funksiyani sodda kasrlarga ajratamiz

$$\frac{1}{x^2-3x+2}=\frac{1}{x-2}-\frac{1}{x-1}.$$

Bu misolni yechish uchun quyidagi binomial qator yoyilmasidan foydalanamiz

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\ldots+\frac{m(m-1)\ldots(m-n+1)}{n!}x^n+\ldots,$$

$$(-1<x<1).$$

Demak,

$$\frac{1}{x-2}=\frac{-1}{2\left(1-\frac{x}{2}\right)}=-\frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{2^2}+\ldots+\frac{x^n}{2^n}+\ldots\right)=$$

$$=-\frac{1}{2}-\frac{x}{2^2}-\frac{x^2}{2^3}-\ldots-\frac{x^n}{2^{n+1}}-\ldots,\quad(-2<x<2)$$

$$\frac{1}{x-1}=-\frac{1}{1-x}=-\left(1+x+x^2+\ldots+x^n+\ldots\right)=$$

$$=-1-x-x^2-\ldots-x^n-\ldots,\quad(-1<x<1).$$

Qatorlarni hadma-had ayirish natijasida quyidagiga ega bo‘lamiz

$$\frac{1}{x^2-3x+2}=\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{2}+\frac{3}{4}x+\ldots+\frac{2^{n+1}-1}{2^{n+1}}x^n+\ldots$$

$$(-1<x<1).$$

4-misol. $f(x)=e^{-x}\sin x$ funksiyani Makloren qatoriga yoying.

Yechish. Ma’lumki

$$e^x=1+x+\frac{1}{2!}x^2+\ldots+\frac{1}{n!}x^n+\ldots,\quad(-\infty<x<\infty),$$

$$\sin x=x-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots,(-\infty<x<\infty).$$

U holda

$$e^{-x}=1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\ldots+(-1)^n\frac{1}{n!}x^n+\ldots.$$

Ko‘paytmani aniqlaymiz

$$f(x)=e^{-x}\sin x=\left(1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\ldots+(-1)^n\frac{1}{n!}x^n+\ldots\right)\times$$

$$\times\left(x-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots\right)=x-x^2+\frac{1}{3}x^3-\frac{3}{40}x^5+\ldots,$$

$$(-\infty < x < \infty).$$

5-misol. $f(x) = \ln\sqrt{\frac{1+3x}{1-x}}$ funksiyani Makloren qatoriga yoying.

Yechish. Ma'lumki

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \qquad (-1 < x < 1).$$

U holda,

$$\ln(1+3x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n \cdot x^n}{n}, \qquad \left(-\frac{1}{3} < x < \frac{1}{3}\right),$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad (-1 < x < 1),$$

$$\ln\sqrt{\frac{1+3x}{1-x}} = \frac{1}{2}(\ln(1+3x) - \ln(1-x)) =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \left((-1)^{n+1} 3^n + 1\right)\frac{x^n}{n}, \qquad \left(-\frac{1}{3} < x < \frac{1}{3}\right).$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2}$$

ko'rinishdagi qator trigonometrik qator deyiladi. Bu yerda a_0, a_n va b_n lar (n=1,2,3,...) o'zgarmas sonlar, qatorning koeffitsiyentlari. Agar (2) qator yaqinlashsa uning yig'indisi davriy 2π bo'lgan funktsiya bo'ladi, ya'ni $f(x)=(x+2\pi)$.

Faraz qilaylik, davri 2π bo'lgan $f(x)$ funksiya berilgan bo'lsin. Qanday shartlar bajarilganda $f(x)$ funksiya uchun berilgan funktsiyaga yaqinlashuvchi trigonometrik qator topish mumkin?

Qatorning Fur'e koeffitsiyentlarini aniqlaymiz. $f(x)$ funksiya trigonometrik qator yig'indisi bo'lsin, ya'ni

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{3}$$

$$\left|\frac{a_0}{2}\right| + |a_1| + |b_1| + |a_2| + |b_2| + \ldots + |a_n| + |b_n| + \ldots \tag{4}$$

yaqinlashuvchi bo'lsa, $f(x)$ funktsiyani $[-\pi;\pi]$ kesmada integrallash mumkin.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} (\int_{-\pi}^{\pi} a_n \cos nxdx + \int_{-\pi}^{\pi} b_n \sin nxdx).$$

Demak, $\int_{-\pi}^{\pi} \frac{a_0}{2} dx = \pi a_0$, $\int_{-\pi}^{\pi} a_n \cos nxdx = a_n \int_{-\pi}^{\pi} \cos nxdx = \frac{a_n \sin nxdx}{n} \Big|_{-\pi}^{\pi} = 0$

$\int_{-\pi}^{\pi} b_n \sin nxdx = b_n \int_{-\pi}^{\rho} \sin nxdx = -b_n \frac{\cos nx}{n} \Big|_{-\pi}^{\pi} = 0.$ Demak, $a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx.$ (3)

Qatorning qolgan koeffitsiyentlarini topish uchun quyidagilarni inobatga olish kerak. Agar n≠k bo‘lsa,

$$\left.\begin{aligned} &\int_{-\pi}^{\pi} \cos nx \cos kxdx = 0, \\ &\int_{-\pi}^{\pi} \cos nx \sin kxdx = 0, \\ &\int_{-\pi}^{\pi} \sin nx \sin kxdx = 0, \end{aligned}\right\} \qquad (A)$$

Agar n=k bo‘lsa,

$$\left.\begin{aligned} &\int_{-\pi}^{\pi} \cos^2 kxdx = \pi, \\ &\int_{-\pi}^{\pi} \sin kx \cos kxdx = 0, \\ &\int_{-\pi}^{\pi} \sin^2 kxdx = \pi, \end{aligned}\right\} \qquad (B)$$

(2) tenglikni coskx ga ko‘paytirib, [-π;π] gacha integrallaymiz.

$$\int_{-\pi}^{\pi} f(x) \cos kxdx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos kxdx + \sum_{n=1}^{\infty} (a_n \int_{-\pi}^{\pi} \cos nx \cos kxdx + b_n \int_{-\pi}^{\pi} \sin nxkxdx) \quad (A)$$

$$\int_{-\pi}^{\pi} \mathrm{f(x)coskxdx} = \mathrm{a_k} \int_{-\pi}^{\pi} \cos^2 \mathrm{kx\,dx} = \mathrm{a_k}\pi \qquad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kxdx \qquad (5)$$

Shuning kabi,

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kxdx \qquad (6)$$

ekanini topamiz.

(4), (5), (6) formulalar bo‘yicha aniqlangan koeffitsiyentlar f (x) funksiyaning Fur’e koeffitsiyentlari deb ataladi. Bunday koeffitsiyenti (1) qator

Fur’e qatori deyiladi.

1-misol. Davri 2π bo‘lgan f(x) davriy funksiya quyidagicha aniqlangan.

F(x) = x: $\pi < x \le \pi$ bu funksiya bo‘lakli va chegaralangan. Demak uni Furbe qatoriga yoyish mumkin

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x df x = \frac{1}{\pi}\Big/_{-\pi}^{\pi}\frac{x^2}{2} = 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx dx = \frac{1}{\pi}\left(x\frac{\sin kx}{k}\right)\Big/_{-\pi}^{\pi} \quad -1\frac{1}{k}\int_{\pi}^{\pi}\sin kx dx = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx dx = \frac{1}{\pi} - \left[-\frac{\cos kx}{<}\right]\Big/_{-\pi}^{\pi} + \frac{1}{k}\int_{\pi}^{\pi}\cos kx dx = (-1)^{x+1}\frac{2}{k}$$

$$J(x) = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots.(-1)^{k+1}\frac{\sin kx}{k} + \ldots.\right]$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad (2)$$

ko‘rinishdagi qator trigonometrik qator deyiladi. Bu yerda a_0, a_n va b_n lar (n=1,2,3,...) o‘zgarmas sonlar, qatorning koeffitsiyentlari. Agar (2) qator yaqinlashsa uning yig’indisi davriy 2π bo‘lgan funktsiya bo‘ladi, ya’ni $f(x)=(x+2\pi)$.

Faraz qilaylik, davri 2π bo‘lgan $f(x)$ funksiya berilgan bo‘lsin. Qanday shartlar bajarilganda $f(x)$ funksiya uchun berilgan funksiyaga yaqinlashuvchi trigonometrik qator topish mumkin?

Qatorning Fur’e koeffitsiyentlarini aniqlaymiz. $f(x)$ funksiya trigonometrik qator yig’indisi bo‘lsin, ya’ni

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{2}$$

$$\left|\frac{a_0}{2}\right|+|a_1|+|b_1|+|a_2|+|b_2|+\ldots+|a_n|+|b_n|+\ldots \tag{3}$$

yaqinlashuvchi bo‘lsa, f (x) funksiyani [-π;π] kesmada integrallash mumkin.

$$\int_{-\pi}^{\pi}f(x)dx=\int_{-\pi}^{\pi}\frac{a_0}{2}dx+\sum_{n=1}^{\infty}(\int_{-\pi}^{\pi}a_n\cos nxdx+\int_{-\pi}^{\pi}b_n\sin nxdx).$$

Demak, $\int_{-\pi}^{\pi}\frac{a_0}{2}dx=\pi a_0$, $\int_{-\pi}^{\pi}a_n\cos nxdx=a_n\int_{-\pi}^{\pi}\cos nxdx=\frac{a_n\sin nxdx}{n}\Big|_{-\pi}^{\pi}=0$

$\int_{-\pi}^{\pi}b_n\sin nxdx=b_n\int_{-\pi}^{\rho}\sin nxdx=-b_n\frac{\cos nx}{n}\Big|_{-\pi}^{\pi}=0.$ Demak, $a_o=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx.$ (4)

Qatorning qolgan koeffitsiyentlarini topish uchun quyidagilarni inobatga olish kerak. Agar n≠k bo‘lsa,

$$\left.\begin{aligned}&\int_{-\pi}^{\pi}\cos nx\cos kxdx=0,\\&\int_{-\pi}^{\pi}\cos nx\sin kxdx=0,\\&\int_{-\pi}^{\pi}\sin nx\sin kxdx=0,\end{aligned}\right\} \quad (A)$$

Agar n=k bo‘lsa,

$$\left.\begin{aligned}&\int_{-\pi}^{\pi}\cos^2 kxdx=\pi,\\&\int_{-\pi}^{\pi}\sin kx\cos kxdx=0,\\&\int_{-\pi}^{\pi}\sin^2 kxdx=\pi,\end{aligned}\right\} \quad (B)$$

(2) tenglikni coskx ga ko‘paytirib, [-π;π] gacha integrallaymiz.

$$\int_{-\pi}^{\pi}f(x)\cos kxdx=\frac{a_0}{2}\int_{-\pi}^{\pi}\cos kxdx+\sum_{n=1}^{\infty}(a_n\int_{-\pi}^{\pi}\cos nx\cos kxdx+b_n\int_{-\pi}^{\pi}\sin nxkxdx) \quad (A)$$

$$\int_{-\pi}^{\pi}\mathrm{f(x)coskxdx}=\mathrm{a_k}\int_{-\pi}^{\pi}\cos^2\mathrm{kx\,dx}=\mathrm{a_k}\pi \qquad \mathrm{a_k}=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kxdx \tag{5}$$

Shuning kabi $b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kxdx$ (6)

ekanini topamiz.

(4), (5), (6) formulalar bo‘yicha aniqlangan koeffitsiyentlar f (x) funksiyaning Fur’e koeffitsiyentlari deb ataladi. Bunday koeffitsiyenti (1) qator Fur’e qatori deyiladi.

Topshiriqlar.

Quyidagi funksiyalarni $(x-x_0)$ ning darajalari bo‘yicha Teylor qatoriga

yoying.

1. $f(x)=\frac{1}{x};\ x_0=3$ Javob: $f(x)=\frac{1}{3}-\frac{x-3}{3^2}+\ldots+(-1)^{n+1}\frac{(x-3)^{n-1}}{3^n}+\ldots, 0<x<6.$

2. $f(x)=x^2-3x;\quad x_0=1$ Javob: $f(x)=-2+3(x-1)^2+(x-1)^3.$

3. $f(x)=x^4-3x^2;\quad x_0=-2$ Javob: $f(x)=-16(x+2)+20(x+2)^2-8(x+2)^3+(x+2)^4.$

4. $f(x)=\ln(x+2);\quad x_0=1$ Javob: $f(x)=\ln 3+\frac{1}{3}(x-1)-\frac{1}{3^2}\frac{(x-1)}{2}+\frac{1}{3^3}\frac{(x-1)}{3}+\ldots+$

$+\frac{(-1)^{n-1}}{3^n}\frac{(x-1)^{n-1}}{n}+\ldots,\ -2\le x\le 4.$

5. $f(x)=\cos x;\quad x_0=\frac{\pi}{4}$ Javob: $f(x)=\frac{\sqrt{2}}{2}\left(1-\frac{x-\frac{\pi}{4}}{1!}-\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}+\ldots\right) -\infty\le x\le+\infty.$

6. $f(x)=e^x;\quad x_0=-2$ Javob: $f(x)=e^{-2}\left(1+\frac{x+2}{1!}-\frac{(x+2)^2}{2!}+\ldots+\frac{(x+2)^n}{n!}+\ldots\right) -\infty\le x\le+\infty.$

Quyidagi funksiyalarni x ning darajalari bo‘yicha Makloren qatoriga yoying.

1. $f(x)=5^x$ Javob: $f(x)=5^x=1+\sum_{n=1}^{\infty}\frac{x^n\ln^n 5}{n!};\quad (-\infty<x<\infty).$

2. $f(x)=\frac{e^x-1}{x}$ Javob: $f(x)=\frac{e^x-1}{x}=1+\frac{x}{2!}+\ldots+\frac{x^{n-1}}{n!}+\ldots,(-\infty<x<\infty).$

3. $f(x)=\ln(10+x)$ Javob: $f(x)=\ln(10+x)=\ln 10+\frac{x}{10}-\frac{x^2}{2\cdot 10^2}+\frac{x^3}{3\cdot 10^3}-\ldots+$

$+(-1)^{n-1}\frac{x^n}{n\cdot 10^n}+\ldots,\ (-\infty<x<\infty).$

4. $f(x)=\sin^2 x$ Javob: $f(x)=\sin^2 x=\frac{2x^2}{2!}-\frac{8x^4}{4!}+\ldots+(-1)^{n-1}\frac{2^{2n-1}x^{2n}}{(2n)!}+\ldots$

$(-\infty<x<\infty).$

5. $f(x)=\sin^2 x\cos^2 x$ Javob: $f(x)=\sin^2 x\cos^2 x=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{4n-3}x^{2n}}{(2n)!},$

$\sin^2 x\cos^2 x=\frac{1}{4}\sin^2 2x,\ (-\infty<x<\infty).$

6. $f(x)=\frac{3}{(1-x)(1+2x)}$ Javob: $f(x)=\frac{3}{(1-x)(1+2x)}=\sum_{n=0}^{\infty}\left(1+(-1)^n 2^{n+1}\right)x^n,$

$\frac{3}{(1-x)(1+2x)}=\frac{1}{1-x}+\frac{2}{1+2x},\ \frac{1}{1-x}=1+x+x^2+\ldots=\sum_{n=0}^{\infty}x^n,$

$$\frac{1}{1+2x}=1-2x+(2x)^2-\ldots+(-1)^n 2^n\cdot x^n+\ldots=\sum_{n=0}^{\infty}(-1)^n 2^n x^n,$$

$$f(x)=\sum_{n=0}^{\infty}x^n+2\sum_{n=0}^{\infty}(-1)^n 2^n x^n=\sum_{n=0}^{\infty}\left(1+(-1)^n 2^{n+1}\right)x^n.$$

7. $f(x)=(1+x)e^{-x}$ Javob: $f(x)=(1+x)e^{-x}=1+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{n-1}{n!}x^n,(-\infty<x<\infty)$.

8. $f(x)=\ln\frac{1+x}{1-x}$ Javob: $f(x)=\ln\frac{1+x}{1-x}=2x+\frac{2x^3}{3}+\ldots+\frac{2x^{2n+1}}{2n+1}+\ldots,(-1<x<1)$

9. $f(x)=\frac{x^2}{\sqrt{1-x^2}}$ Javob: $f(x)=\frac{x^2}{\sqrt{1-x^2}}=x^2+\frac{1}{2}x^2+\ldots+\frac{(2n-1)!!}{(2n)!!}x^{2n-2}$,

$(-1<x<1))$

10. $f(x)=\frac{\ln(1+x)}{1+x}$ Javob: $f(x)=\frac{\ln(1+x)}{1+x}=x-\left(1+\frac{1}{2}\right)x^2+\left(1+\frac{1}{2}+\frac{1}{3}\right)x^3-$

$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)x^4+\ldots,(-1<x<1)$

11. $f(x)=\ln(1+3x+2x^2)$ Javob: $f(x)=\ln(1+3x+2x^2)=(-1)^{n-1}(1+2^n)\frac{x^n}{n},\left(-\frac{1}{2}<x\le\frac{1}{2}\right)$

12. Binomial qator yordami bilan $|x|<1$ bo‘lganda

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1\cdot3}{2\cdot4}x^4+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6+\ldots+$$

$+\frac{1\cdot3\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot\ldots\cdot(2n)}x^{2n}+\ldots$ ekanini ko‘rsating va qatorni hadma-had integrallab, $\arcsin x$ uchun qatorga yoying. Javob:

$$\arcsin x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\frac{x^5}{5}+\ldots+\frac{1\cdot3\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot\ldots\cdot(2n)}\frac{x^{2n+1}}{2n+1}+\ldots.$$

13. Binomial qator yordami bilan $|x|<1$ bo‘lganda

$$\frac{1}{\sqrt{1+x^2}}=1-\frac{1}{2}x^2+\frac{1\cdot3}{2^2\cdot2!}x^4-\frac{1\cdot3\cdot5}{2^3\cdot3!}x^6+\ldots$$

ekanligini ko‘rsating va qatorni hadma-had integrallab, $\ln\left(x+\sqrt{1+x^2}\right)$ funksiya uchun qatorga yoying.

Quyidagi funksiyalarni Fur’e qatorga yoyish.

1) $f(x) = \begin{cases} A, agar\ 0 < x < l \\ 0, agar\ l < x < 2l \end{cases}$ $(0,2l)$ intervalda Fur'e qatoriga yoying.

2) $f(x) = \frac{\pi - x}{2}$ $(0,2\pi)$ intervalda Fur'e qatoriga yoying.

3) $f(x) = |x|$ $(-\pi,\pi)$ intervalda Fur'e qatoriga yoying.

4) $f(x) = \pi^2 - x^2$ $(-\pi,\pi)$ intervalda Fur'e qatoriga yoying.

5) $f(x) = cos2x$ $(-\pi,\pi)$ intervalda Fur'e qatoriga yoying.

6) $f(x) = x$ $(0,2l)$ intervalda Fur'e qatoriga yoying.

7) $f(x) = xsinx$ $(-\pi,\pi)$ intervalda Fur'e qatoriga yoying.

8) $f(x) = xcosx$ $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ intervalda Fur'e qatoriga yoying.

10-MAVZU. IKKI O‘LCHOVLI INTEGRALNI HISOBLASH. IKKI O‘LCHOVLI INTEGRALLARNING GEOMETRIYA VA MEXANIKAGA TADBIQLARIGA DOIR MASHQLAR

Asosi G sohadan iborat, yasovchilari Oz o‘qiga parallel, tenglamasi $z = f(x,y) \geq 0$ bo‘lgan silindrik sirtning hajmini topish masalasini qaraymiz. Buning uchun G sohani n ta $\Delta\sigma,...,\Delta\sigma n$ yuzalarga ajratib, har bir yuzachaning $P_i(x_i, y_i)$ nuqtalarining z_i applikatasini olamiz. $z_i = f(x_i, y_i)$ ni $\Delta\sigma_i$ yuziga ko‘paytirib, barcha shunday ko‘paytmalarning yig’indisini olamiz:

$V = \sum_{i=1}^{n} f(x_i,\ y_i)\ \Delta 6_i$ - bu integral yig’indi deyiladi. Integral yig’indining $n \to \infty$ dagi limiti $z = f(x,y)$ funksiyadan D soha bo‘yicha olingan ikki o‘lchovli integral deyiladi:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,\ y_i)\ \Delta\sigma_i = \iint f(x,\ y)\ \Delta\sigma = \iint f(x,\ y)\, d\sigma;$$

Ikki o‘lchovli integralni Dekart koordinatalar sistemasida hisoblash.

$a \leq x \leq b;\ \ f_1(x) \leq y \leq f_2(x)$ bo‘lsa, $\iint\limits_G f(x,\ y)\, dx\, dy = \int\limits_a^b \left(\int\limits_{f_1(x)}^{f_2(x)} f(x,\ y)\, dx \right) dx$ bo‘ladi.

1-misol. $f(x,\ y) = x^2 + y \sin xy$ funksiyadan $G: \{0 \leq x \leq 1;\ 0 \leq y \leq 1\}$ soha bo‘yicha ikki o‘lchovli integralni hisoblaylik.

Yechish: $\iint\limits_G (x^2 + y\sin xy)\, dx\, dy = \int\limits_0^1 \left(\int\limits_0^1 (x^2 + y\sin xy)\, dx \right) dy = \int\limits_0^1 \left(\frac{x^3}{3} - \cos xy \right)\Big|_0^1 dy$

$= \left(\frac{1}{3} y - \sin y + y \right)\Big|_0^1 = \frac{4}{3} - \sin 1$

2-misol: $f(x,\ y) = xe^y + y\cos x$ funksyadan $G: \{0 \leq x \leq 1;\ 0 \leq y \leq 1\}$ to‘rtburchakli soha bo‘yicha ikki o‘lchovli integralni hisoblaylik

Yechish: $\iint\limits_G f(x,y)\, dx\, dy = \int\limits_0^{\pi} \left(\int\limits_0^1 (xe^y + y\cos x)\, dy \right) dx = \int\limits_0^{\pi} \left(xe^y + \frac{y^2}{2}\sin x \right)\Big|_0^1 dx =$

$$= \int_0^{\pi} \left(xe + \frac{1}{2}\cos x - x \right) dx = \left(\frac{x^2}{2} e - \sin x - \frac{x^2}{2} \right) \Big|_0^{\pi} = \frac{\pi^2}{2}(e-1);$$

Topshiriq.

1. $\iint (xy + y^2) dx\, dy,$ integralni y=x, y=2x, x=1 chiziqlar bilan chegaralangan G soha bo‘yicha hisoblaylik.

2. $f(x,y) = (x - y)$ funksiyani D: $\{1 \le x \le 3,\ y = x, y = x^3\}$ soha bo‘yicha integralini hisoblang:

3. To‘g’ri to‘rtburchakli sohada quyidagi integrallarni hisoblang.

$\iint_D xydxdy$ *bu yerda* $(0 \le x \le 1,\quad 0 \le y \le 2)$;

4. $\iint_D x^2 y \cos(xy^2) dxdy; \qquad 0 \le x \le \frac{\pi}{2}; \qquad 0 \le y \le 2$

5. $\iint_D f(x,y) dx\, dy \quad D: 0 \le x \le 1, 0 \le y \le 2$

6. $x + y \le 1, x - y \le 1, x \ge 0$
$y \le x - 1, y \ge 1 - x$
$\int_0^1 \left(\int f(x,y) dy \right) dx;$

$\frac{x^2}{4} + \frac{y^2}{9} \le 1;$

7. $\frac{y^2}{9} \le 1 - \frac{x^2}{4};$
$\frac{|y|}{3} \le \frac{1}{2}\sqrt{4 - x^2}$
$-\frac{3}{2}\sqrt{4 - 2x} \le y \le \frac{3}{2}\sqrt{4 - x^2};$

Quyidagi integrallarning tartibini o‘zgartiring

1. $\int_0^z dx \int_x^{\sqrt{2>x-x^2}} f(x,y) dy;$ 2. $\int_1^2 dx \int_x^{2x} f(x,y) dy$

3. $\int_0^1 dx \int_0^x f(x,y) dy + \int_1^2 dx \int_0^{2-x} f(x,y) dy$

4. $\int_0^1 dx \int_0^{x^2} f(x,y)dy + \int_1^3 \left(\int_0^{\frac{(3-x)}{2}} f(x,y)dy \right) dx$

5. $\int_0^1 dx \int_0^{x^2} f(x,y)dy + \int_1^3 dx \int_0^{\frac{1}{2}(3-x)} f(x,y)dy$

Quyidagi integrallarni hisoblang:

1. $\int_0^a dx \int_0^{\sqrt{x}} dy$

2. $\iint_D \left(x^2 + y\right)dxdy = \qquad y = x^2, \qquad y^2 = x$

3. $\int_0^2 dx \int_{2x}^{6-x} f(x,y)dy \qquad x = 0, x = 2 \qquad y = 2x, \qquad y = 6 - x$

4. $\iint_D \cos(x+y)dxdy \qquad x = 0, \ y = \pi, \qquad y = x;$

5. $\iint_D x^2 y^2 \sqrt{1 - x^3 - y^3}\, dxdy; x = 0, y = 0, x^3 + y^3 = 1$

6. $z = x + 6y \qquad y = x, \qquad y = 5x, \qquad x = 1$

7. $z = \sqrt{R^2 - x^2 - y^2}; x^2 + y^2 \le R^2 \qquad \rho = R, x = R$

$y = \pm\sqrt{R^2 - x^2}; \begin{cases} x = \rho\cos\varphi \\ y = \sin\varphi \end{cases} \qquad -R = R\cos\varphi, \varphi = \pi$

$R = R\cos\varphi \pm \varphi = 0$

8. Koordinatalar tekisliklari x=4 va y=4 tekisliklar va $z = x^2 + y^2 + 1$ paraboloid chegaralangan sirt hajmini toping.

9. Tekisliklar $x = a;\ y = b;\ z = \frac{x^2}{2p} + \frac{y^2}{2q}$ elliptik paraboloid.

10. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad z = 0 \ \ da \ \ \frac{x}{a} + \frac{y}{b} = 1 \qquad y = b - \frac{b}{a}x;$

$x = a; \ y = b; \ z = c \qquad z = c \cdot \left(1 - \frac{x}{a} - \frac{y}{b}\right);$

11. $y=0,\ z=0,\ 3x+y=6,\ 3x+2y=12$ va $x+y+z=6$, tekisliklar bilan chegaralangan jism hajmini toping.

$$0\le y\le 6 \qquad 2-\frac{1}{3}y\le x\le 4-\frac{2}{3}y;\quad x=2-\frac{1}{3}y \quad 0\le z\le 6-x-y \quad x=4-\frac{2}{3}y$$

12. $z=x^2+y^2;\ z=0;\ y=0;\ x=0;\ x+y=1;\ y=1-x$.

13. $(x^2+y^2)^2=2a^2(x^2-y^2)$ Bernulli lemniskati

$$\begin{array}{ll} x=\rho\ \cos\varphi & \rho^2=a^2\ \cos 2\ \varphi \\ y=\rho\ \sin\varphi & \rho=a\ \sqrt{\cos 2\varphi} \end{array} \qquad 0\le\varphi\le\frac{\pi}{4};$$

14. $y=x;\ y=2a;\ x=1$ tekisliklar va $z=x^2+y^2;\ z=x^2+2y^2$ paraboloidlar bilan chegaralanga jism hajmini toping.

11-MAVZU. UCH O‘LCHOVLI INTEGRALNI HISOBLASH. UCH O‘LCHOVLI INTEGRALNING TADBIQLARIGA DOIR MASHQLAR

1-xossa. Agar $V=V_1+V_2$ bo‘lsa, u holda $\mathbf{J_V=J_{V1}+J_{V2}}$ (1) , bo‘ladi.

Natija. Agar $V = V_1 + V_2 + ... + V_n$ bo‘lsa, u holda,

$$J_v = J_{v_1} + J_{v_2} + ... + J_{v_n} \qquad (2)$$

2-xossa. (Uch karrali integralni baholash)

Agar m va M lar $f(x,y,z)$ funksiyani V dagi eng kichik va eng katta qiymatlari bo‘lsa, u holda $m \cdot V \le J_v \le M \cdot V$ bo‘ladi, bunda V uch o‘lchovli sohani hajmi.

3-xossa. (O‘rta qiymat haqidagi teorema)

$f(x,y,z)$ funksiyada V bo‘yicha olingan uch karrali J_V integral, shu integrallash soha hajmini $f(x,y,z)$ funksiyani shu sohadan olingan P nuqtadagi qiymatiga ko‘paytirilganiga teng, ya’ni $J_V= f$ (p)V

Teorema. $f(x,y,z)$ sohadan V soha bo‘yicha olingan uch o‘lchovli integral, shu soha bo‘yicha olingan uch karrali integralga teng, ya’ni

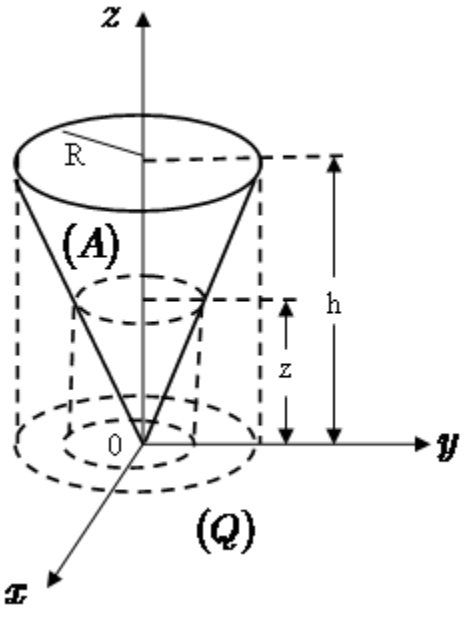
$$\iiint_V f(x,y,z)dv = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz \right) dy \right) dx$$

Misol.

Ushbu

$$I = \iiint_{(A)} z dx dy dz$$

integral hisoblansin. Bu yerda (A) jism $z^2 = \frac{h^2}{R^2}\left(x^2+y^2\right)$ konusli sirt va $z=h$ tekislik bilan chegaralangan soha.



(a) konusli sirtni xy tekislikdagi proyeksiyasi

(Q) $x^2+y^2\le R^2$ doiradan iborat. ushbu formuladan foydalanib

$$I=\iint\limits_{(Q)}dxdy\int\limits_{\frac{h}{R}\sqrt{x^2+y^2}}^{h}zdz=\frac{1}{2}\iint\limits_{(Q)}\left[h^2-\frac{h^2}{R^2}\left(x^2+y^2\right)\right]dxdy$$

bo‘ladi. Bundan qutb koordinatalariga o‘tib,

$$I=\frac{h^2}{2R^2}\int\limits_0^{2\pi}d\theta\int\limits_0^R\left(R^2-r^2\right)rdr=\frac{\pi R^2h^2}{4}.$$

(b) boshqa usulda integralning qiymatini

$$I=\int\limits_0^h zdz\iint\limits_{(D)}dxdy$$

ko‘rinishda ham hisoblab topish mumkin. Bu yerda (D) xy tekislikdagi proyeksiyasi bilan balandlik yotgan z tekislikning kesishmasi. Bu proyeksiya $\frac{Rz}{h}$ radiusli doiradan iborat. Shuningdek ikki karrali integral doiraning yuzi $\frac{\pi R^2}{h^2}z^2$ ga teng. Bundan

$$I=\int\limits_0^h\frac{\pi R^2}{h^2}z^3dz=\frac{\pi R^2h^2}{4}$$

bo‘lishini topamiz.

Topshiriqlar.

1.Uch karrali integrallarni hisoblang.

1)$\iiint_V xy^2z^3dxdydz,$ bu yerda V $z=xy,y=x,x=1,z=0$ sirtlar bilan chegaralangan soha.

2)$\iiint_V \frac{dxdydz}{(1+x+y+z)^3}$ bu yerda V $x+y+z=1,x=0,y=0,z=0$ sirtlar bilan chegaralangan soha.

3)$\iiint_V xyzdxdydz,$ bu yerda V $x^2+y^2+z^2=1,x=0,y=0,z=0$ sirtlar bilan chegaralangan soha.

4)$\iiint_V \sqrt{x^2+y^2}\,dxdydz$, $bu\ yerda\ V\ x^2+y^2=z^2,z=1$ sirtlar bilan chegaralangan soha.

2. Uch karrali integral yordamida quyidagi sirtlar bilan chegralangan jismlarning hajmini hisoblang.

1)$z = x^2 + y^2, z = 2x^2 + 2y^2, y = x, y = x^2$

2)$z = x + y, z = xy, x + y = 1, x = 0, y = 0.$

3)$x^2 + z^2 = a^2, x + y = \pm a, x - y = \pm a$

4)$z = 6 - x^2 - y^2, z = \sqrt{x^2 + y^2}$

3. Uch karrali integrallarning mexanikaga tatbiqlari.

1)Agar jismning $M(x, y, z)$ nuqtasidagi zichligi $\rho = x + y + z$ bo‘lib, u hajmi 1 ga teng bo‘lgan $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ kubni egallagan bo‘lsa, uning hajmini toping.

2) Agar jism cheksiz $x^2 + y^2 + z^2 \ge 1$ sohani egallagan bo‘lib, uning har bir nuqtasida zichligi $\rho = \rho_0 e^{-k\sqrt{x^2+y^2+z^2}}, \rho_0 > 0 \; va \; k > 0$ qonun bilan o‘zgarsa, uning massasini hisoblang.

Quyidagi sirtlar bilan chegaralangan bir jinsli jismning massa markazini toping.

1). $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}, \; z = c$.

2). $z = x^2 + y^2, x + y = a, x = 0, y = 0, z = 0.$

3). $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, x = 0, y = 0, z = 0$.

4)$x^2 + y^2 = 2z, \; x + y = z$

4. Quyidagi sirtlar bilan chegaralangan bir jinsli jismlarning koordinata tekisliklariga nisbatan inersiya momentlarini hisoblang.(parametrlar musbat aniqlangan)

1). $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, x = 0, y = 0, z = 0$

2). $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

3). $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}, \; z = c$

$1) zichligi \; \rho = z \; qonun \; bilan \; o'zgaradigan \; z = \frac{x^2+y^2}{2} \; paraboloidning \; (0 \le z \le 1) qismidagi \; massasini$
hisoblang.

2)Har bir $M(x, y, z)$ nuqtasida zichligi $\frac{z}{a}$ ga teng bo‘lgan $x^2 + y^2 + z^2 = a^2,$

$z \geq 0$ yarim sferaning massasini hisoblang.

3)$x + y + z = a \quad (x \geq 0, y \geq 0, z \geq 0)$ bir jinsli uchburchakning koordinata tekisliklariga nisbatan stastik momentlarini toping.

4) $x + y + z = 1 \quad (x \geq 0, y \geq 0, z \geq 0)$ bir jinsli uchburchakning kordinata tekisliklariga nisbatan inersiya momentlarini toping.

5)$z = \sqrt{a^2 - x^2 - y^2}$ $(x \geq 0, y \geq 0, x + y \leq a)$bir jinsli sirtning massa markazi koordinatalarini hisoblang.

FOYDALANILGAN ADABIYOTLAR RO‘YXATI

1. Кремер Н.Ш. Теория вероятностей и математическая статистика. – М.: ЮНИТИ, 2001. -330 стр.

2. Колемаев В.А., Калинина В.Н. Теория вероятностей и математическая статистика. Учебник. - М.: Инфра-М., 1997. -350 стр.

3. Гмурман В.Е. Руководство к решению задач по теории вероятностей и математической статистике. Учебное пособие. Москва. “Высшая школа” 2004. – 407 стр.

4. Бабаджанов Ш. Ш. “Теория вероятностей и математическая статистика”. Курс лекций. Ташкент, ТМИ, 2004. 152 стр.

5. Azlarov T., Abdushukurov J. Ehtimollar nazariyasi va matematik statistikadan masalalar to‘plami. -T.: 2000. -180 bet.

6. Muxitdinov T., Sagdullayev D., Shoraxmetov Sh. Ehtimollar nazariyasi va matematik statistika fanidan masalalar to‘plami. -T.: 2005. -150 bet.

7. Mamurov E.N., Adirov T.X. “Ehtimollar nazariyasi va matematik statistika”.- T.:TMI. 2005. -135 bet.

8. Сборник задач по курсу “Математика в экономике”, в трех частях, “Теория вероятностей”. Учебное пособие под редакцией, Бабайцева В.А., Гисина В.Б. -М.: ИНФРА -М. 2010. 128 стр.

9. Adirov T. X., Hamdamov I. “Ehtimollar nazariyasi va matematik statistikadan masalalar va ularni yechishga oid ko‘rsatmalar”. Toshkent. “Iqtisod-moliya”, 2008. 120 bet.

10. Бабаджанов Ш. Ш. “Материалы самостоятельных работ по Теория вероятностей и математическая статистика”. Учебное пособие. Ташкент, ТМИ, 2006. 90 стр.

11. Sharipov Sh. R. va Mo‘minov N.S. “Oddiy differensial tenglamalar” Toshkent. 1986 y.

12. Филиппов А. Ф “Сборник задач по дифференциальным уравнениям” Москва. “Наука” 1985 г.